Elmer W. Boehmer '29
721 Monroe St. Phone 79

DIFFERENTIAL AND INTEGRAL CALCULUS

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO · DALLAS
ATLANTA · SAN FRANCISCO

MACMILLAN & CO., LIMITED
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN CO. OF CANADA, LTD.
TORONTO

DIFFERENTIAL AND INTEGRAL

CALCULUS

BY

CLYDE E. LOVE, PH.D.

PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF MICHIGAN

REVISED EDITION

New York

THE MACMILLAN COMPANY

1926

All rights reserved

PRINTED IN THE UNITED STATES OF AMERICA

COPYRIGHT, 1916 AND 1925,
BY THE MACMILLAN COMPANY.

Set up and electrotyped. Published September, 1916.
Revised edition, July, 1925. Reprinted October, 1925;
April, 1926; December, 1926.

L. H. JENKINS, INC.
RICHMOND, VIRGINIA

PREFACE TO THE REVISED EDITION

The revised edition of the author's "Differential and Integral Calculus" shows no radical departure from the original, but it is hoped that the book has been materially improved in various respects. Many articles have been rewritten, several topics of minor importance have been omitted, and some additions have been made, chiefly in the chapters on transcendental functions, indefinite integration, and rectilinear motion. The exercises are nearly twice as numerous as in the original edition; they are also more varied, and better graded. The number of worked examples has likewise been increased.

The author is under deep obligation to Professor A. L. Nelson, who has read the entire manuscript with great care and has also helped throughout the proof-reading. He is also indebted to Professors T. H. Hildebrandt and L. J. Rouse for a number of useful suggestions.

CLYDE E. LOVE.

ANN ARBOR,
May, 1925.

PREFACE TO THE FIRST EDITION

This book presents a first course in the calculus substantially as the author has taught it at the University of Michigan for a number of years. The following points may be mentioned as more or less prominent features of the book.

In the treatment of each topic, the text is intended to contain a precise statement of the fundamental principle involved, and to insure the student's clear understanding of this principle, without distracting his attention by the discussion of a multitude of details. The accompanying exercises are intended to present the problem in hand in a great variety of forms and guises, and to train the student in adapting the general methods of the text to fit these various forms. The constant aim is to prevent the work from degenerating into mere mechanical routine, as it so often tends to do. Wherever possible, except in the purely formal parts of the course, the summarizing of the theory into rules or formulas which can be applied blindly has been avoided. For instance, in the chapter on geometric applications of the definite integral, stress is laid on the fact that the basic formulas are those of elementary geometry, and special formulas involving a coördinate system are omitted.

Where the passage from theory to practice would be too difficult for the average student, worked examples are inserted.

It seems clear that so-called applications in which the student is made to use a formula without explanation of its meaning and derivation, are of little value. In the present text the non-geometric applications are taken systematically from one subject, mechanics, and the theory is developed as fully as in the calculus proper.

A feature of the book is its insistence on the importance of checking the results of exercises, either directly or by solving in more than one way. The latter method is largely used in the integral calculus, on account of the variety of elementary transformations possible with definite integrals.

The answers to many of the exercises are given, but seldom where a knowledge of the answer would help in the solution, or where a simple means of checking the answer exists.

Topics of minor importance are presented in such a way that they may be omitted if it is desired to give a short course.

The chapter on curve tracing is introduced as early as possible, so that the results are available for use throughout the course.

Some instructors will wish to begin the use of integral tables immediately after the chapters on formal integration. This of course can easily be done.

In spite of obvious difficulties, a chapter embodying a first treatment of centroids and moments of inertia is introduced before multiple integrals have been defined. By this arrangement the student is brought to realize the fact that in most cases of practical importance mass-moments of the first and second orders can be found by simple integration, whereas from the usual treatment he gets exactly the opposite idea.

In the chapters on differential equations, emphasis is laid on those types most likely to be met by the student of engineering or the mathematical sciences. In the last chapter the average student will doubtless require considerable help from the instructor, but it is hoped that, if properly presented, the chapter may give the student some facility in writing and solving the simpler differential equations of mechanics and in interpreting the results.

To Professor Alexander Ziwet, who has read the entire manuscript, the author makes grateful acknowledgment, not only for valuable advice and criticism, but for his un-

failing encouragement and support. Thanks are also due to Professor T. H. Hildebrandt, who has kindly assisted in reading the proofs, and has made a number of useful suggestions.

CLYDE E. LOVE.

ANN ARBOR,
August, 1916.

CONTENTS

CHAPTER I

FUNCTIONS. LIMITS. CONTINUITY

ART.		PAGE
1.	Functions	1
2.	Graphic representation	3
3.	Classification of functions	3
4.	One-valued and many-valued functions	4
5.	Rate of change; slope	6
6.	Limits	6
7.	Limit of a function	7
8.	Theorems on limits	8
9.	Continuity	10
10.	Infinity	11
11.	Function with infinite argument	12

CHAPTER II

THE DERIVATIVE

12.	The derivative	14

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

13.	Introduction	19
14.	Derivative of a constant	19
15.	Derivative of a sum; a product; a quotient	19
16.	Derivative of x^n, n a positive integer	21
17.	Derivative of a function of a function	25
18.	The general power formula	25
19.	Higher derivatives	26
20.	Implicit functions	29
21.	Differentiation of implicit functions	30

CHAPTER IV

GEOMETRIC APPLICATIONS

ART.		PAGE
22.	Tangents and normals to plane curves	33
23.	Graphs	36
24.	Preliminary steps	36
25.	Increasing and decreasing functions	37
26.	Maxima and minima	37
27.	Concavity	38
28.	Points of inflection	40
29.	Summary	40
30.	Applications of the theory of maxima and minima	43
31.	A variation of the method	44
32.	Derived curves	47

CHAPTER V

TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

I. Trigonometric Functions

33.	Elementary properties	50
34.	Differentiation of $\sin x$	50
35.	Limit of $\sin \alpha/\alpha$ as α approaches 0	51
36.	Differentiation of $\cos x$, $\tan x$, etc.	52
37.	Graphs of trigonometric functions	55
38.	Curve tracing by composition of ordinates	56
39.	Graphic solution of equations	57
40.	Maxima and minima	58

II. Inverse Trigonometric Functions

41.	Inverse functions	60
42.	Inverse trigonometric functions	61
43.	Restriction to a single branch	62
44.	Inverse trigonometric equations	62
45.	Differentiation of the inverse trigonometric functions	65

CHAPTER VI

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

46.	The exponential function	68
47.	The logarithm	69
48.	Fundamental properties of logarithms	70

ART. PAGE
49. Change of base 70
50. Natural logarithms 71
51. Exponential and logarithmic equations 74
52. The derivative of the logarithm 76
53. The power formula 78
54. Differentiation of the exponential function 79
55. Hyperbolic functions 81

CHAPTER VII

THE DIFFERENTIAL

56. Infinitesimals 85
57. Order of infinitesimals 85
58. Principal part of an infinitesimal 86
59. The differential 89
60. Approximate formulas 92
61. Parametric equations 96

CHAPTER VIII

CURVATURE

62. Differential of arc 99
63. Curvature 100
64. Radius of curvature 101

CHAPTER IX

TIME-RATES. MOTION

65. Miscellaneous applications 104
66. Velocity and acceleration in rectilinear motion . . . 107
67. Vectors 110
68. Velocity in curvilinear motion 110
69. Acceleration in curvilinear motion 111
70. Rotation 113

CHAPTER X

INDETERMINATE FORMS

71. The Law of the Mean 115
72. The indeterminate forms $0/0$, ∞/∞ 116
73. The indeterminate form $0 \cdot \infty$ 118

CHAPTER XI

CURVE TRACING

ART.		PAGE
74.	Horizontal and vertical asymptotes	120
75.	Restriction of the curve to definite regions	121
76.	Curve tracing	121
77.	One-valued algebraic functions	122
78.	Two-valued algebraic functions	124
79.	Transcendental curves	125
80.	Singular points	126
81.	Determination of tangents by inspection	127
82.	Classification of double points	129
83.	Curve tracing in polar coördinates	131

CHAPTER XII

THE INDEFINITE INTEGRAL

84.	Integration	132
85.	Integration an indirect process	134
86.	Constant of integration	134
87.	Functions having the same derivative	135

CHAPTER XIII

STANDARD FORMULAS OF INTEGRATION

88.	Standard formulas	136
89.	Formulas (1)–(3)	137
90.	Formula (4): Powers	137
91.	Formulas (5)–(6): Logarithms and exponentials	140
92.	Formulas (7)–(9): Trigonometric functions	142
93.	Formulas (10)–(11): Inverse trigonometric functions	143
94.	Formula (12): Integration by parts	143

CHAPTER XIV

TRIGONOMETRIC INTEGRALS

95.	Introduction	148
96.	Transformation by the identities of trigonometry	148
97.	Transformation by multiple angles	151

CHAPTER XV

INTEGRATION BY SUBSTITUTION

ART. PAGE
98. Integration by substitution 152
99. Trigonometric substitutions 155

CHAPTER XVI

INTEGRATION OF RATIONAL FRACTIONS

100. Preliminary step 157
101. Partial fractions 157
102. Distinct linear factors 158
103. Repeated linear factors 160
104. Quadratic factors 162

CHAPTER XVII

THE DEFINITE INTEGRAL

105. The definite integral 165
106. Interchanging limits 166
107. Change of limits corresponding to a change of variable . 167
108. Integrand containing two or more variables . . . 168
109. Parametric equations 169
110. The integral interpreted as an area 170
111. Improper integrals 172

CHAPTER XVIII

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

112. Area under a curve 176
113. Evaluation of the limit 178
114. The fundamental theorem 178
115. Plane areas in cartesian coördinates 179
116. Plane areas in polar coördinates 183
117. Solids of revolution 184
118. Solids of revolution: second method 186
119. A theorem on infinitesimals 187
120. Miscellaneous solids 191
121. Length of a curvilinear arc 195

ART. PAGE
122. Surfaces of revolution 197
123. Force 199
124. Fluid pressure 200

CHAPTER XIX

CENTROIDS

125. Density 203
126. Moment of mass 203
127. Centroid 204
128. Determination of centroids by integration 208
129. Centroids of plane areas 209
130. Centroids of solids 212
131. Centroids of lines; of curved surfaces 214

CHAPTER XX

MOMENTS OF INERTIA

132. Moment of inertia 215
133. Radius of gyration 215
134. Determination of moment of inertia by integration . . 216
135. Moment of inertia with respect to a plane 221
136. General theorems on moments of inertia 223
137. Kinetic energy of a rotating body 225

CHAPTER XXI

SERIES OF CONSTANT TERMS

138. Series of n terms 228
139. Infinite series 228
140. Sum of an infinite series 229
141. Convergence and divergence 230
142. Integral test 232
143. Comparison test 234
144. Ratio test 236
145. Alternating series 238
146. Absolute convergence 239
147. Evaluation of the sum of a series 240
148. Limit of error 240

CHAPTER XXII

POWER SERIES

ART.		PAGE
149.	Power series	243
150.	Maclaurin's series	246
151.	Applications of Maclaurin's series	248
152.	The value of e	249
153.	Operations with power series	252
154.	The value of π	256
155.	Computation of logarithms	257
156.	Taylor's series	261
157.	Approximate formulas for Δy	262

CHAPTER XXIII

PARTIAL DIFFERENTIATION

158.	Functions of several variables	265
159.	Limits; continuity	266
160.	Partial derivatives	266
161.	Geometric interpretation of partial derivatives	267
162.	Higher derivatives	267
163.	Total differentials	269
164.	Approximate formulas	270
165.	Differentiation of implicit functions	270
166.	Tangent plane and normal line	274
167.	Angle between two surfaces; between a line and a surface	276

CHAPTER XXIV

MULTIPLE INTEGRALS

168.	Volume under a surface	279
169.	The double integral in polar coördinates	284
170.	Solids of revolution: polar coördinates	287
171.	Evaluation by inversion of order	288
172.	Evaluation by change of coördinate system	289
173.	Heterogeneous masses	290
174.	Triple integrals	294

CHAPTER XXV

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

175.	General definition	298
176.	Order of a differential equation	298
177.	Solutions of a differential equation	299

ART. PAGE
178. The general solution 299
179. Particular solutions 301
180. Separation of variables 302
181. Coefficients homogeneous of the same degree . . . 305
182. Exact differential equations 306
183. Integrating factors 307
184. The linear equation 309
185. Equations linear in $f(y)$ 310

CHAPTER XXVI

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

186. General and particular solutions 313
187. The equation $y'' = f(x)$ 315
188. Dependent variable absent 315
189. Independent variable absent 316
190. The linear equation 318
191. The homogeneous linear equation 318
192. The characteristic equation 319
193. Distinct roots 319
194. Equal roots 321
195. The imaginary exponential function 321
196. Complex roots 322
197. Extension to equations of higher order 325
198. The non-homogeneous linear equation 327

CHAPTER XXVII

RECTILINEAR MOTION

199. Introduction 331
200. Motion of a particle under given forces 332
201. The equation of motion 332
202. Uniformly accelerated motion 333
203. Atwood's machine 334
204. Motion in a resisting medium 338
205. Simple harmonic motion 342
206. Hooke's law 343
207. Damped vibrations 346
208. Momentum 350
209. Impulse 350
210. The principle of impulse and momentum 351

ART. PAGE
211. Kinetic energy 352
212. Work 352
213. The principle of kinetic energy and work 353

CHAPTER XXVIII

PLANE CURVILINEAR MOTION

214. Rotation 355
215. The simple pendulum 355
216. The equations of motion 357
217. Projectiles 357
218. Central force proportional to the distance 359
Indefinite integrals 361

CALCULUS

CHAPTER I

FUNCTIONS. LIMITS. CONTINUITY

1. Functions. If a variable y depends upon a variable x so that to every value of x there corresponds a value of y, then y is said to be a *function of* x.

For example, (a) the area of a circle is determined by the radius and is therefore a function of the radius; (b) the attraction (or repulsion) between two magnetic poles is a function of the distance between them; (c) the volume of a given mass of gas at a constant temperature is a function of the pressure upon the gas.

A complete study of the properties of a function is possible in general only when the function is given by a definite mathematical expression. For this reason we shall be concerned almost entirely with functions defined in this way. Thus, in the examples above, we have

(a) $A = \pi r^2$,

(b) $F = \dfrac{k}{r^2}$,

(c) for a "perfect gas,"

$$v = \frac{k}{p}.$$

But the existence of a functional relation between two quantities does not imply the possibility of giving this relation a mathematical formulation. If by any means whatever a value of y is determined corresponding to every value of x under consideration, then y is a function of x. For example, the temperature of the air at any point of the earth's surface is a function of the time at which the ther-

mometer is read, although no mathematical law connecting the two variables is known.

We often wish to express merely the fact that y is a function of x, without assigning the particular form of the function. This is done by writing

$$y = f(x)$$

(read y equals f of x). Other letters may of course be used in the functional symbol, as $F(x)$, $\phi(x)$, $\psi(x)$, etc.

The value of $f(x)$ when $x = a$ is denoted by the symbol $f(a)$. Thus, if

$$f(x) = x^2 - 3x - 1,$$

then

$$f(a) = a^2 - 3a - 1,$$
$$f(2) = -3,$$
$$f(x + h) = (x + h)^2 - 3(x + h) - 1.$$

Except where the contrary is explicitly stated, the variables and functions with which we shall have to deal are restricted to real values. This restriction is introduced for the sake of simplicity, and also because in the elementary applications only real quantities are of importance.

We usually think of x as varying arbitrarily — *i.e.* we assign values to x at pleasure, and compute the corresponding values of y. The variable x is then called the *independent variable*, or *argument*. But it is clear that if y is a function of x, x is likewise a function of y, and in general either one may be chosen as the independent variable.

The values assigned to x must of course be compatible with the conditions of the problem in hand. In most cases x is restricted to a definite *range* or *interval;* for instance, if the function in question is

$$y = \sqrt{x},$$

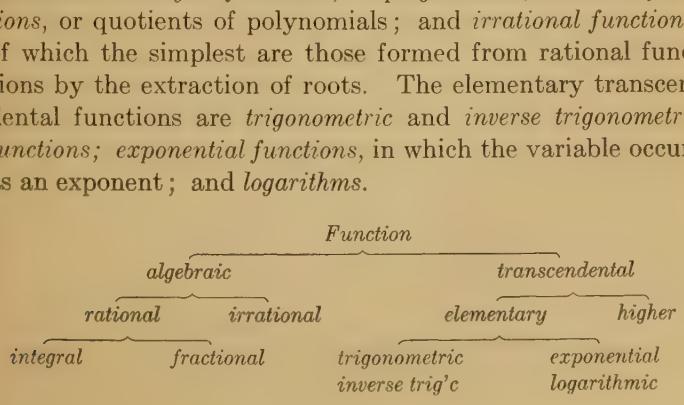
then x is limited to positive values, since when x is negative, y is imaginary. Again, in example (*a*), negative values of the radius are excluded because they would have no geometric meaning.

2. Graphic representation. The equation $y = f(x)$ determines a value of y corresponding to every value of x under consideration. The pairs of values of x and y thus determined may be considered as the cartesian coördinates of points in a plane; the curve made up of these points is called the *graph* of the function. Graphic representation of a function is frequently very helpful in studying its properties.

It is clear that the graph of the function $f(x)$ is identical with the locus of the equation $y = f(x)$ as defined in analytic geometry. Thus in (*a*) of § 1 the graph is a parabola; in (*b*) it is a certain cubic curve; in (*c*) it is a hyperbola.

Even though no mathematical expression for the function is known, it may still be represented graphically. For instance, to represent the temperature at a point of the earth's surface as a function of the time, let a large number of readings be taken, the corresponding points be plotted on coördinate paper with time as abscissa and temperature as ordinate, and a smooth curve be drawn through the points. This curve will represent approximately the variation of temperature throughout the time-interval in question.

3. Classification of functions. All functions are classed as either *algebraic* or *transcendental*. The algebraic functions are *rational integral functions*, or *polynomials; rational fractions*, or quotients of polynomials; and *irrational functions*, of which the simplest are those formed from rational functions by the extraction of roots. The elementary transcendental functions are *trigonometric* and *inverse trigonometric functions; exponential functions*, in which the variable occurs as an exponent; and *logarithms*.



4. One-valued and many-valued functions. A function $y = f(x)$ is said to be *one-valued*, if to every value of x corresponds a single value of y; *two-valued*, if to every value of x correspond two values of y, etc.

In the case of a many-valued function it is usual to group the values in such a way as to form a number of one-valued functions, called the *branches* of the original function. Thus the equation

$$y^2 = x$$

defines a two-valued function whose branches are

$$y = \sqrt{x},$$
$$y = -\sqrt{x}.$$

In dealing with many-valued functions, we shall in general *confine our attention to a particular branch.*

EXERCISES

1. Mention various examples of functional relationship in addition to those of the text; where possible, express each one by a formula.

2. Find $\phi(0)$, $\phi(1)$, $\phi(2)$, $\phi(-1)$, $\phi(-2)$, if
$$\phi(x) = x^3 - 3x^2 + 5x - 1.$$

3. Find $f(0), f(1), f(2), f(3), f(4)$, if
$$f(x) = x^4 - 3x^3 + x^2 - x.$$

4. Find $\phi(0)$, $\phi(\frac{1}{2})$, $\phi(-\frac{1}{2})$, if
$$\phi(x) = x(1 - 4x^2).$$

5. Find $F(0)$, $F(2)$, $F(-1)$, $F(x+2)$, if

(*a*) $F(x) = 3x^2 - 5x + 6$; (*b*) $F(x) = \dfrac{1-2x}{1+2x^2}$;

(*c*) $F(\theta) = \frac{1}{2}\theta^3$; (*d*) $F(y) = \sqrt{y^2+5}$;

(*e*) $F(z) = \dfrac{\sqrt{z^2-4}}{\sqrt{2z-5}}$; (*f*) $F(x) = (x-2)^3$.

6. Find $f(0)$, $f(\frac{1}{4}\pi)$, $f(\pi)$, $f(x+2\pi)$, if

(*a*) $f(x) = \sin x - \cos x$; (*b*) $f(x) = \tan^2 x$;

(*c*) $f(x) = \sin^2 2x$; (*d*) $f(x) = \sin \frac{1}{2}x + \cos \frac{1}{2}x$.

7. Find $f(1)$, $f(\frac{1}{2})$, $f(-\frac{3}{2})$, $f(x+2)$, if

(*a*) $f(x) = 2^x$; (*b*) $f(x) = 10^{1-x}$.

8. Classify the functions of Exs. 2, 3, 4, 5, 6, 7.

9. Represent graphically the functions (*a*), (*b*), (*c*) of § 1.

In each of the following cases, draw the graph of the function.

10. The distance covered by a man walking at 3 miles per hour, as a function of the time.

11. The surface and volume of a cube as functions of the length of its edge.

12. The acceleration j produced by a constant force F, as a function of the mass moved:

$$j = \frac{F}{m}.$$

13. The amount of a sum of money at 5% simple interest, as a function of the time.

14. The resistance of air to the motion of a projectile, as a function of the velocity, using the approximate formula $R = kv^{\frac{3}{2}}$.

15. The distance between two magnetic poles, as a function of the force between them:

$$d = \frac{k}{\sqrt{F}}.$$

16. The sum of an infinite geometric series, as a function of the ratio:

$$S = \frac{a}{1 - r},$$

where a is the first term. (The formula holds for $-1 < r < 1$.)

17. The sum of the geometric progression

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}},$$

as a function of the number of terms: $S = 2 - \frac{1}{2^{n-1}}$. What portion of the graph has a meaning in this case?

18. If $f(x) = \cos x$, find $f(-x), f(\pi - x), f(2\pi - x)$.

19. If $f(x) = \sin x$, find $f(2x), f(-x), f(\pi - x)$.

20. If $f(x) = a^x$, show that $f(x + y) = f(x) \cdot f(y)$.

21. Show that the equation $x^2 + y^2 = a^2$ defines y as a two-valued function of x, and draw the graph.

Plot the graphs of the functions defined by the following equations.

22. $y^2 = a^2 + x^2$.

23. $y^2 = \frac{x^2}{1 + x^2}$.

24. $y^2 = x(x - 2)^2$.

25. $y^2 = x^3$.

26. $y^2 = x^2 - a^2$.

27. $y^2 = x^2(1 + x^2)$.

28. Show that (*a*) the graph of a one-valued function is met by any parallel to the y-axis in not more than one point; (*b*) the graph of a many-valued function consists of a number of branches (not necessarily disconnected), each of which has this same property.

5. Rate of change; slope. A fundamental problem in studying the nature of a function is the determination of its *rate of change.*

Given y as a function of x, let us assign to x an arbitrary change, or *increment*, Δx (read delta x), usually taken positive, and denote by Δy the corresponding change (positive or negative) in y. The ratio $\frac{\Delta y}{\Delta x}$ is the *average rate of change* of y as x changes from its initial value to the value $x + \Delta x$. If now we let Δx approach 0, the ratio $\frac{\Delta y}{\Delta x}$ in general approaches a definite limiting value, which is defined as the *rate of change* of y corresponding to the given value of x.

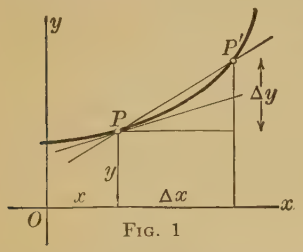


FIG. 1

The geometric interpretation is obvious. The point $P:(x, y)$ is a point on the curve

$$y = f(x);$$

the point $P':(x + \Delta x,\ y + \Delta y)$ is a neighboring point on the curve. The ratio $\frac{\Delta y}{\Delta x}$ is the slope of the secant PP'. When Δx is taken smaller and smaller, the point P' approaches P along the curve, the secant PP' approaches the tangent at P as its limiting position, and $\frac{\Delta y}{\Delta x}$ approaches as its limit the slope of the tangent. Hence *the rate of change of a function is the slope of its graph.*

6. Limits. From what has just been said, it appears that the determination of the rate of change of a function, or the slope of a curve, requires the evaluation of a certain *limit.* It will therefore be well to introduce at this point a brief discussion of the subject of limits.

When the successive values of a variable x approach nearer and nearer a fixed number a, in such a way that the

difference $a - x$ *becomes* and *remains* numerically less than any preassigned positive number however small, the constant a is called the *limit* of x, and x is said to *approach the limit* a — in symbols,

$$\lim x = a.$$

Examples are easily found in elementary work:

(*a*) If a regular polygon be inscribed in a circle, the difference between the area A_P of the polygon and the area A_C of the circle becomes arbitrarily small (less than any preassigned positive number) as the number of sides increases indefinitely. Hence

$$\lim A_P = A_C.$$

(*b*) We know from elementary algebra that the sum S_n of the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

is

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

The difference between 2 and S_n is

$$2 - S_n = \frac{1}{2^{n-1}}.$$

This difference becomes arbitrarily small as the number of terms increases indefinitely; hence

$$\lim S_n = 2.$$

7. Limit of a function. We have frequently to observe the behavior of a function $f(x)$ as the argument x approaches a limit. If, as x approaches a, the difference between $f(x)$ and some fixed number l ultimately becomes and remains numerically less than any preassigned positive constant however small, the function $f(x)$ is said to *approach the limit* l, and we write

$$\lim_{x \to a} f(x) = l.$$

Unless otherwise specified it is supposed that the same limit is approached whether x comes toward a from the positive

or the negative direction. If we wish to consider what happens when x approaches a from the positive side only, we write $\lim_{x \to a^+} f(x)$; from the negative side only, $\lim_{x \to a^-} f(x)$.

8. Theorems on limits. We shall have occasion to use the following theorems on limits, which we assume without formal proof.

THEOREM I*: *The limit of the sum of two variables is equal to the sum of their limits.*

THEOREM II: *The limit of the product of two variables is equal to the product of their limits.*

THEOREM III: *The limit of the quotient of two variables is equal to the quotient of their limits, provided the limit of the denominator is not* 0.

Theorems I and II may evidently be extended to the case of any number of variables.

The exceptional case of Theorem III, in which the denominator approaches 0, requires further investigation. Given any fraction $\frac{v}{u}$ in which u approaches 0, two cases are to be distinguished:

(a) v also approaches 0;

(b) v does not approach 0.

It is clear that in case (b) the fraction $\frac{v}{u}$ may be made to assume values greater than any assignable constant by taking u sufficiently small; hence the fraction can approach no limit. But consider (a), where the numerator and the denominator both approach 0. Theorem III does not apply; the ratio of the limits is $\frac{0}{0}$, which is quite meaningless; nevertheless the *limit of the ratio* may exist, as will appear

* In these theorems it is of course implied that the limits of the two variables exist. We shall see later (§ 72) that the quotient of two variables, for instance, may approach a limit when neither of the two variables taken by itself approaches a limit.

in many cases in the next few chapters. In fact, it will appear as we proceed that the entire differential calculus is based on the determination of limits of this type.

EXERCISES

Evaluate the following limits.

1. $\lim_{x \to 3} (1 - 6x - 6x^2)$.

2. $\lim_{y \to -2} (y^3 - 3y^2 + y)$.

3. $\lim_{t \to 0} (3t^4 - 5t + 2)$.

4. $\lim_{x \to \frac{1}{2}} (3x^3 - 3x^2 + 2)$.

5. $\lim_{t \to 1} \dfrac{t^3 - 5t^2 + 4}{t^3 + 1}$.

6. $\lim_{x \to 0} \left(\dfrac{x^2 - 5x + 6}{x^2 - 5}\right)^2$.

7. $\lim_{x \to -3} \dfrac{x^2 + 5x}{x^2 - 5x}$.

8. $\lim_{y \to \frac{1}{3}} \dfrac{9y^2 - 3y}{y^3 + 2y - 1}$.

9. $\lim_{\theta \to \frac{\pi}{3}} \dfrac{\sin\theta \cos 2\theta}{\tan \frac{1}{2}\theta}$.

10. $\lim_{\alpha \to \frac{\pi}{4}} \dfrac{\cot\alpha - \cos\alpha}{\sin 2\alpha - \sin\alpha}$.

11. $\lim_{x \to 0} x\, 10^{-x}$. *Ans.* 0.

12. $\lim_{x \to \pi} x^2 \cos x$.

13. $\lim_{y \to 3} \dfrac{y^2 - 9}{2y^2 - 6y}$. *Ans.* 1.

14. $\lim_{u \to 2} \dfrac{\sqrt{6 - 3u}}{\sqrt{4 - u^2}}$. *Ans.* $\frac{1}{2}\sqrt{3}$.

15. $\lim_{x \to -2} \dfrac{x^3 + 8}{x^2 + 5x + 6}$. *Ans.* 12.

16. $\lim_{x \to -\frac{1}{2}} \dfrac{4x^2 + 4x + 1}{2x^3 + x^2}$. *Ans.* 0.

17. $\lim_{x \to 1} \dfrac{1 - x}{\sqrt{1 - x^2}}$. *Ans.* 0.

18. $\lim_{x \to 1} \dfrac{\sqrt{1 - x^3}}{\sqrt{1 - x^2}}$. *Ans.* $\frac{1}{2}\sqrt{6}$.

19. $\lim_{x \to 1} \dfrac{2x^2 + 3x - 5}{3x^2 - 2x - 1}$. *Ans.* $\frac{7}{4}$.

20. $\lim_{x \to 3} \dfrac{3x^2 - 13x + 12}{2x^2 - 3x - 9}$. *Ans.* $\frac{5}{9}$.

21. $\lim_{x \to 0} \dfrac{\tan x}{\sin x}$. *Ans.* 1.

22. $\lim_{x \to 0} \dfrac{\tan x}{\sin 2x}$. *Ans.* $\frac{1}{2}$.

23. $\lim_{\theta \to 0} \dfrac{\tan 2\theta}{\sin\theta}$. *Ans.* 2.

24. $\lim_{\theta \to \frac{\pi}{2}} \dfrac{\cos 2\theta}{\tan\theta}$. *Ans.* 0.

25. $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{(x - 2)^2}$. *Ans.* No limit.

26. $\lim_{\theta \to 0} \dfrac{\sin 2\theta}{\sin^2\theta}$. *Ans.* No limit.

27. $\lim_{\theta \to 0} \dfrac{\sin^2\theta}{1 - \cos\theta}$.

28. $\lim_{\theta \to \pi} \dfrac{\sin^2\theta}{1 + \cos\theta}$.

29. Show that, if n is a positive integer,

$$\lim x^n = (\lim x)^n.$$

30. Show that, if $P(x)$ is a polynomial in x,

$$\lim_{x \to a} P(x) = P(a).$$

31. Show that, if $P_1(x)$ and $P_2(x)$ are polynomials,

$$\lim_{x \to a} \frac{P_1(x)}{P_2(x)} = \frac{P_1(a)}{P_2(a)},$$

provided $P_2(a) \neq 0$.

32. Under what circumstances may the limit in Ex. 31 exist when $P_2(a) = 0$? Give an example.

33. Does the limit in Ex. 31 always exist when $P_1(a) = P_2(a) = 0$? Give examples.

9. Continuity. An important idea in the study of functions is that of *continuity.*

A function $f(x)$ is said to be *continuous* at the point $x = a$ if it approaches a definite limit as x approaches a, and if the limit so approached is the value of the function at that point; *i.e.* if

$$\lim_{x \to a} f(x) = f(a).$$

A function is said to be *continuous in an interval* of values of the argument if it is continuous at all points of the interval.

As x changes from any value a to any other value b, the function $f(x)$, if continuous in the interval $x = a$ to $x = b$, assumes every value intermediate between $f(a)$ and $f(b)$*. The graph of the function is without any gap or break in the interval in question; it is " continuous " in the ordinary sense of the word.

In the discussion of § 5, it is tacitly assumed that the function is continuous in an interval including the point P; this assumption is an essential part of the argument.

All the functions treated in this book are continuous, except perhaps for certain particular values of the variable, and such values are either excluded or subjected to special investigation.

* This theorem is almost self-evident; a strict proof is beyond the scope of this book.

Examples: (*a*) The function

$$y = \frac{1}{x - 2}$$

is continuous for all values of x except $x = 2$. As x approaches 2, the function increases indefinitely (see Fig. 2 below).

(*b*) The function

$$y = \cos \frac{\pi}{x}$$

is continuous for all values of x except 0. As x approaches 0, the function oscillates between -1 and 1 without approaching any limit.

10. Infinity. The most important type of discontinuity is that in which the function increases numerically without limit, or, as we say, *becomes infinite*, as x approaches a. In this case we write

$$\lim_{x \to a} f(x) = \infty.$$

But it must be noted that this equation is merely symbolic, for the reason that *the symbol ∞ does not represent a number.* The symbolic equation tells us, not that $f(x)$ approaches some vague, indefinite, very large limiting value, but that it increases beyond any limit whatever.

Graphically the occurrence of such a discontinuity means that the curve $y = f(x)$ approaches nearer and nearer the line $x = a$, usually without ever reaching it, at the same time receding indefinitely from the x-axis.

Example: As x approaches 2, the function

$$y = \frac{1}{x - 2}$$

becomes infinite:

$$\lim_{x \to 2} \frac{1}{x - 2} = \infty.$$

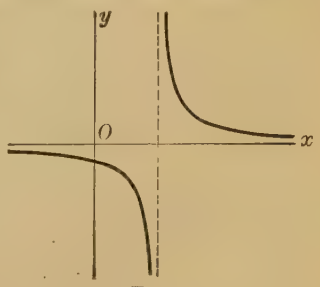


FIG. 2

The graph of the function is shown in Fig. 2.

11. Function with infinite argument. We have frequently to investigate the behavior of a function as the argument becomes infinite.

If, when x increases without limit, the difference between $f(x)$ and some fixed number l ultimately becomes and remains numerically less than any preassigned positive constant however small, we write

$$\lim_{x\to\infty} f(x) = l.$$

Graphically this means that the curve $y = f(x)$ approaches nearer and nearer the line $y = l$, usually without ever reaching it, at the same time receding indefinitely from the y-axis.

Examples: (*a*) To evaluate

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1},$$

we divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x\to\infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 1.$$

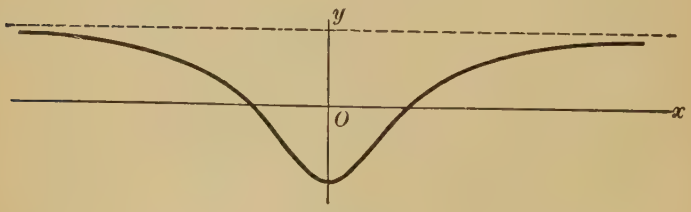


FIG. 3

The graph of the function

$$y = \frac{x^2 - 1}{x^2 + 1}$$

is shown in Fig. 3.

(*b*) $$\lim_{x\to\infty} \frac{x^2}{x + 1} = \lim_{x\to\infty} \frac{x}{1 + \frac{1}{x}} = \infty.$$

Thus in this case no limit is approached.

EXERCISES

1. Show that a polynomial is continuous for all values of x (see Ex. 30, p. 10).

2. Show that a rational fraction is continuous for all values of x except those for which the denominator is 0 (see Ex. 31, p. 10).

Find the points of discontinuity of the following functions.

3. $\dfrac{1}{x^2-9}$. *Ans.* $x = \pm 3$.

4. $\dfrac{x^2}{x+5}$. *Ans.* $x = -5$.

5. $\dfrac{x^2-3}{x^2-5x+4}$.

6. $\dfrac{1}{2x^2+3x+1}$.

7. $\dfrac{x^2+3x+2}{x^3+6x^2+8x}$.

8. $\dfrac{x^3}{2x^2-3x-5}$.

9. $\dfrac{1}{x^3-6x^2-2x}$.

10. $\dfrac{x^2-1}{x^2+2x+3}$. *Ans.* None.

Evaluate the following limits.

11. $\lim\limits_{x\to\infty} \dfrac{1}{x-3}$. *Ans.* 0.

12. $\lim\limits_{x\to\infty} \dfrac{x+1}{x-1}$. *Ans.* 1.

13. $\lim\limits_{x\to\infty} \dfrac{x^2+3x}{x^2-x+1}$.

14. $\lim\limits_{x\to\infty} \dfrac{2x+3}{3x-5}$.

15. $\lim\limits_{x\to\infty} \dfrac{x^2-2x}{3x^3-1}$. *Ans.* 0.

16. $\lim\limits_{x\to\infty} \dfrac{3x^3}{x^3+x^2-1}$.

17. $\lim\limits_{x\to\infty} \dfrac{x^2+3x}{x-1}$.

18. $\lim\limits_{x\to\infty} \dfrac{3x^4}{x^2-5}$.

19. $\lim\limits_{x\to\infty} \dfrac{(2x-1)^2}{x(x+3)}$. *Ans.* 4.

20. $\lim\limits_{x\to\infty} \dfrac{(3x-1)^3}{(x-3)^3}$.

21. $\lim\limits_{x\to-\infty} 10^x$. *Ans.* 0.

22. $\lim\limits_{x\to\infty} 10^x$.

23. $\lim\limits_{x\to\infty} \tan x$. *Ans.* No limit.

24. Does $\sin x$ approach any limit as x becomes infinite? Does $\dfrac{\sin x}{x}$? Does $\dfrac{\tan x}{x}$?

25. Show that as x approaches 0, the function $\sin\dfrac{\pi}{x}$ oscillates between -1 and 1, without approaching any limit.

26. Discuss the behavior of $\tan\dfrac{\pi}{x}$ near the origin.

27. Discuss the behavior of $10^{\frac{1}{x}}$ near the origin.

28. If $f(x)$ is continuous, is its square continuous? Is its reciprocal?

29. Given two continuous functions, what can be said of the continuity of their sum? Their product? Their quotient?

CHAPTER II

THE DERIVATIVE

12. The derivative. Given a function

$$y = f(x),$$

continuous at the point $P: (x, y)$, let us assign to x an arbitrary increment Δx, and compute the corresponding increment Δy of y. We have

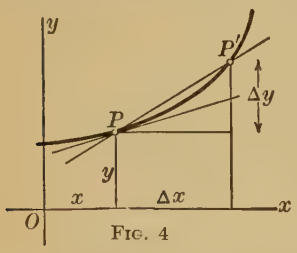


FIG. 4

$$y + \Delta y = f(x + \Delta x),$$

so that

$$\Delta y = f(x + \Delta x) - f(x).$$

Now form the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The limit of the ratio $\frac{\Delta y}{\Delta x}$ *as* Δx *approaches* 0 *is called the* ***derivative*** *of* y *with respect to* x.

The derivative is designated by the symbol $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Other symbols for the derivative are* y', $f'(x)$.

The operation of finding the derivative is called *differentiation.*

* Since the symbol y' does not explicitly indicate the independent variable, its use should be restricted to the case in which y is a function of x, so that always $y' = \frac{dy}{dx}$. If $y = f(t)$, say, the derivative should be denoted by $\frac{dy}{dt}$, or $f'(t)$.

It follows from § 5 that *the derivative of a function is identical with its rate of change. Geometrically the derivative of a function is the slope of its graph.*

Only differentiable functions (*i.e.* those having a derivative) *are considered in this book.* In some cases the derivative may fail to exist for particular values of the argument, but such values are either excluded or subjected to special investigation.

To find the derivative of a given function, we have merely to *build up the "difference-quotient"* $\frac{\Delta y}{\Delta x}$ *and then pass to the limit as* Δx *approaches* 0. It will be remembered that this is essentially the method used in analytic geometry to find the slope of a curve. Since Δx and Δy approach 0 together, our problem is to find the limit of a fraction whose numerator and denominator both approach 0 (cf. § 8). In general, this limit cannot be evaluated until some suitable transformation, algebraic or otherwise, has been applied to the quotient $\frac{\Delta y}{\Delta x}$.

The process of finding the derivative is illustrated by the following

Examples: (*a*) Find the slope of the parabola

$$y = 2x^2 - 6x + 4$$

at the point (x, y); at the point (2, 0).

If

$$y = f(x) = 2x^2 - 6x + 4,$$

then

$$\begin{aligned} y + \Delta y &= f(x + \Delta x) \\ &= 2(x + \Delta x)^2 - 6(x + \Delta x) + 4, \\ \Delta y &= 4x\Delta x + 2\overline{\Delta x}^2 - 6\Delta x, \\ \frac{\Delta y}{\Delta x} &= 4x + 2\Delta x - 6, \\ y' &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4x - 6. \end{aligned}$$

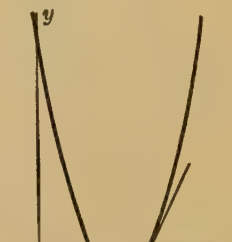


FIG. 5

Hence the slope at any point (x, y) is $4x - 6$; in particular, the slope at the point (2, 0) is 2.

(*b*) If the distance of a moving body from a fixed point of reference, at any time t, is

$$x = \frac{1}{t+1},$$

find the rate of change of x.

If

$$x = \frac{1}{t+1},$$

then

$$x + \Delta x = \frac{1}{t + \Delta t + 1},$$

$$\Delta x = \frac{1}{t + \Delta t + 1} - \frac{1}{t+1}.$$

Reducing the fractions in the right member to a common denominator, we find

$$\Delta x = \frac{t + 1 - (t + \Delta t + 1)}{(t + \Delta t + 1)(t+1)}$$

$$= \frac{-\Delta t}{(t + \Delta t + 1)(t+1)}.$$

Whence

$$\frac{\Delta x}{\Delta t} = \frac{-1}{(t + \Delta t + 1)(t+1)},$$

and

$$\frac{dx}{dt} = \frac{-1}{(t+1)^2}.$$

(*c*) Differentiate the function $y = \sqrt{x}$.

If

$$y = \sqrt{x},$$

then

$$y + \Delta y = \sqrt{x + \Delta x},$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= (\sqrt{x + \Delta x} - \sqrt{x}) \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{(x + \Delta x) - x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}},$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}},$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

EXERCISES

Differentiate the following functions.

1. $y = 2x - 3x^2$. *Ans.* $y' = 2 - 6x$.

2. $y = 3 - 4x - \frac{1}{2}x^2$. *Ans.* $y' = -4 - x$.

3. $s = t^3$. *Ans.* $\frac{ds}{dt} = 3t^2$.

4. $y = (2t - 3)^2$. *Ans.* $\frac{dy}{dt} = 4(2t - 3)$.

5. $x = 3z^3 - 5z^2 - 2z + 7$.

6. $r = \theta^4 - 3\theta^3$.

7. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 6x - 5$.

8. $y = (x - a)^3$.

9. $y = \frac{1}{x}$. *Ans.* $y' = -\frac{1}{x^2}$.

10. $y = \frac{1}{2x - 3}$. *Ans.* $y' = \frac{-2}{(2x - 3)^2}$.

11. $y = \frac{x}{1 - x}$. *Ans.* $y' = \frac{1}{(1 - x)^2}$.

12. $s = \frac{t^2}{1 - t}$. *Ans.* $\frac{ds}{dt} = \frac{2t - t^2}{(1 - t)^2}$.

13. $y = \frac{1}{x^2}$.

14. $y = \frac{3x - 4}{2x - 5}$.

15. $y = \sqrt{x + 1}$.

16. $y = \sqrt{3 - x}$.

17. $y = \sqrt{2 - 3x}$.

18. $x = \frac{1}{2}\sqrt{5t + 6}$.

19. $y = \frac{1}{\sqrt{x}}$. *Ans.* $y' = \frac{-1}{2x^{\frac{3}{2}}}$.

20. $u = v^{\frac{3}{2}}$. *Ans.* $\frac{du}{dv} = \frac{3}{2}v^{\frac{1}{2}}$.

21. $y = \frac{1}{\sqrt{1 - x}}$. *Ans.* $y' = \frac{1}{2(1 - x)^{\frac{3}{2}}}$.

22. $y = \sqrt{1 - x^2}$. *Ans.* $y' = \frac{-x}{\sqrt{1 - x^2}}$.

23. $x = \frac{1}{\sqrt{2t + 1}}$. *Ans.* $\frac{dx}{dt} = \frac{-1}{(2t + 1)^{\frac{3}{2}}}$.

24. $x = \frac{1}{\sqrt{1 + t^2}}$. *Ans.* $\frac{dx}{dt} = \frac{-t}{(1 + t^2)^{\frac{3}{2}}}$.

In the following cases, find the slope of the curve at the point indicated.

25. $y = 3x^2 - 5x + 6, \quad x = 2.$ *Ans.* 7.

26. $y = x^3 - 3x^2 + 4x - 1, \quad x = -2.$

27. $y = \dfrac{1}{2 - 3x}, \quad x = 0.$ *Ans.* $\frac{3}{4}$.

28. $y^2 = 4ax, \quad (a, 2a).$

29. $y^2 + 3x - 4 = 0, \quad (1, -1).$ *Ans.* $\frac{3}{2}$.

In the following cases, if x is measured in feet and t in seconds, find the rate of change of x at the time given.

30. $x = 3t^2 - 4t - 5, \quad t = 0.$ *Ans.* -4 ft. per sec.

31. $x = \sqrt{t + 2}, \quad t = 1.$ *Ans.* $\frac{1}{6}\sqrt{3}$ ft. per sec.

32. $x = \dfrac{4 - 2t}{1 + 2t}, \quad t = 2.$ *Ans.* $-\frac{2}{5}$ ft. per sec.

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

13. Introduction. In this and later chapters (V, VI) we develop certain *standard formulas* by means of which any elementary function may be differentiated. The use of these formulas effects a great saving of time, and obviates the necessity of evaluating a special limit in every problem.

The formulas of §§ 15, 17 are direct consequences of the definition of the derivative, and are valid for all functions (*i.e.* all functions that are *continuous, one-valued,* and *differentiable;* see §§ 9, 4, 12).

14. Derivative of a constant. We note first that *the derivative of a constant is* 0:

$$\frac{dc}{dx} = 0. \tag{1}$$

For, if $y = c$, then no matter what the values of x and Δx may be, y will remain unchanged, and hence $\Delta y = 0$:

$$\frac{\Delta y}{\Delta x} = 0, \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

The line $y = c$ is parallel to Ox; its slope is everywhere 0.

15. Derivative of a sum; a product; a quotient. If u and v are functions of x, the following formulas are true by the definition of the derivative:

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}, \tag{2}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \tag{3}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{4}$$

These formulas may be stated in words as follows:

(**2**) *The derivative of the sum of two functions is equal to the sum of their derivatives.*

(**3**) *The derivative of the product of two functions is equal to the first function times the derivative of the second plus the second times the derivative of the first.*

(**4**) *The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, divided by the square of the denominator.*

Proof of (**2**): Let x assume an increment Δx, and denote by Δu and Δv the corresponding increments of u and v. Then

$$\begin{aligned} y &= u + v, \\ y + \Delta y &= u + \Delta u + v + \Delta v, \\ \Delta y &= \Delta u + \Delta v, \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}, \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{du}{dx} + \frac{dv}{dx}. \end{aligned}$$

Proof of (**3**):

$$\begin{aligned} y &= uv, \\ y + \Delta y &= (u + \Delta u)(v + \Delta v), \\ \Delta y &= u\Delta v + v\Delta u + \Delta u \Delta v, \\ \frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}, \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u\frac{dv}{dx} + v\frac{du}{dx}. \end{aligned}$$

Proof of (**4**):

$$\begin{aligned} y &= \frac{u}{v}, \\ y + \Delta y &= \frac{u + \Delta u}{v + \Delta v}, \\ \Delta y &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - uv - u\Delta v}{(v + \Delta v)v}, \end{aligned}$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v + \Delta v)v},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Formulas (**2**) and (**3**) can be extended to the case where n functions are involved. For three functions, (**3**) becomes

$$\frac{d}{dx}uvw = vw\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}.$$

In the special case when $u = c$, a constant, (**3**) and (**4**) become

(**3′**) $$\frac{d}{dx}cv = c\frac{dv}{dx},$$

(**4′**) $$\frac{d}{dx}\frac{c}{v} = -\frac{c\dfrac{dv}{dx}}{v^2}.$$

16. Derivative of x^n, n a positive integer. If

$$y = x^n,$$

where n is a positive integer, then

(1) $$\frac{dy}{dx} = nx^{n-1}.$$

For,

$$y + \Delta y = (x + \Delta x)^n$$

$$= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n,$$

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n,$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + \overline{\Delta x}^{n-1},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

In particular, if $n = 1$, *i.e.* if $y = x$,

$$\frac{dx}{dx} = 1,$$

which is obvious geometrically.

Although the above proof is valid only for positive integral values of n, formula (1) is true for all values of the exponent. The general proof will be given in § 53.

Examples: (*a*) The derivative of

$$y = 2x^3 - 5x^2 + 3x + 2$$

is

$$y' = 6x^2 - 10x + 3.$$

(*b*) The derivative of

$$y = \frac{x^2 + 1}{4x + 3}$$

is

$$y' = \frac{(4x + 3)2x - (x^2 + 1)4}{(4x + 3)^2} = \frac{4x^2 + 6x - 4}{(4x + 3)^2}.$$

(*c*) To differentiate

$$y = \frac{1}{\sqrt{x}},$$

we write

$$y = x^{-\frac{1}{2}},$$

whence

$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x^{\frac{3}{2}}}.$$

EXERCISES

Differentiate the following functions.

1. $y = 7x^3 - 3x^2 + 8x + 11.$
2. $y = 1 - 3x - 5x^2 - x^3.$
3. $y = 2x^4 - 3x^2 + x - 5.$
4. $y = x - 3x^3 - 2x^5.$
5. $y = 2x^{\frac{1}{3}} - 3x^{\frac{4}{3}}.$
6. $y = 3\sqrt{x^3} - 2\sqrt{x} + \sqrt{3}.$
7. $y = \frac{2}{5}x^{-2} - \frac{3}{4}x^{-\frac{1}{2}} + x^{\frac{1}{2}}.$
8. $y = \frac{x^{-1}}{2} - \frac{2x^{-4}}{3} + \frac{1}{2}x^{-\frac{5}{2}}.$

9. $y = \frac{1}{x} - \frac{1}{2x^2} + \frac{3}{x^3}$.

10. $y = \frac{x^2}{3} - \frac{5x}{2} + \frac{3}{4} - \frac{3}{x} - \frac{2}{x^2}$.

11. $y = \frac{1}{2}\sqrt{x} - \frac{1}{3\sqrt{x}} + \frac{1}{2\sqrt{x^5}}$.

12. $y = \frac{2}{3x^2} - \frac{3}{4x^3} + \frac{2}{7}$.

13. $x = \frac{1}{3}t^3 - 2\sqrt{t} - \frac{1}{t} + \frac{2}{3t^2}$.

14. $u = 4v^6 - \frac{3v^5}{2} + \frac{2v^2}{3} - \frac{2v}{5}$.

15. $y = (x^2 - 3)(2x^2 - 5x + 6)$. *Ans.* $y' = 8x^3 - 15x^2 + 15$.

16. $y = (3x^3 - 2x)(x^2 - 2x - 1)$.

Ans. $y' = 15x^4 - 24x^3 - 15x^2 + 8x + 2$.

17. $y = (3x - 2)(5 - 6x)$.

18. $y = (t^3 - 2)(t^3 - 3t)$.

19. $y = \frac{x}{x-1}$. *Ans.* $y' = \frac{-1}{(x-1)^2}$.

20. $y = \frac{3x-1}{2-2x}$. *Ans.* $y' = \frac{1}{(1-x)^2}$.

21. $y = \frac{t^2}{t^2-2}$. *Ans.* $\frac{dy}{dt} = \frac{-4t}{(t^2-2)^2}$.

22. $x = \frac{y^3 - 2y}{y+1}$. *Ans.* $\frac{dx}{dy} = \frac{2y^3 + 3y^2 - 2}{(y+1)^2}$.

23. $w = \frac{z^3}{z^3-1}$. *Ans.* $\frac{dw}{dz} = \frac{-3z^2}{(z^3-1)^2}$.

24. $y = \frac{1 + 2z - z^2}{z^2+3}$. *Ans.* $\frac{dy}{dz} = \frac{6 - 8z - 2z^2}{(z^2+3)^2}$.

25. $f(x) = \frac{(1+x)(1-2x)}{x}$. *Ans.* $f'(x) = -\frac{1+2x^2}{x^2}$.

26. $f(t) = \frac{t^3}{t-1}$. *Ans.* $f'(t) = \frac{2t^3 - 3t^2}{(t-1)^2}$.

27. $F(x) = \frac{x^2}{1-x^2}$.

28. $\phi(u) = \frac{u^3 - 3u}{2u - 1}$.

Perform the indicated differentiation.

29. $\frac{d}{dx}\left(\frac{4x^5}{5} - 3x^3 + \frac{2x^2}{3} - \frac{x}{2}\right)$.

30. $\frac{d}{dx}\frac{1}{x^2+x-1}$. *Ans.* $-\frac{2x+1}{(x^2+x-1)^2}$.

31. $\frac{d}{dt}\frac{3}{1-t^3}$. *Ans.* $\frac{9t^2}{(1-t^3)^2}$.

32. $\frac{d}{dx}\frac{x^2+3}{x}$. *Ans.* $1 - \frac{3}{x^2}$.

33. $\frac{d}{dz}\frac{a+2z}{a-2z}$. *Ans.* $\frac{4a}{(a-2z)^2}$.

34. $\frac{d}{dy}(y+2)(y^2-3y-1)$. *Ans.* $3y^2-2y-7$.

35. $\frac{d}{du}(u^2+3u)(u^2-5u+1)$.

36. $\frac{d}{dx}\frac{x-1}{x^2-3x-1}$. *Ans.* $-\frac{x^2-2x+4}{(x^2-3x-1)^2}$.

37. $\frac{d}{dy}\frac{y^2+3y+4}{y^2}$. *Ans.* $-\frac{3y+8}{y^3}$.

38. $\frac{d}{dq}\frac{q^2+8q-3}{2q^2-3q-1}$. *Ans.* $-\frac{19q^2-10q+17}{(2q^2-3q-1)^2}$.

Find the slope of each of the following curves at the point indicated.

39. $y=\frac{x^3}{2}-\frac{x^2}{3}+\frac{3x}{5}-\frac{1}{3}$, $x=2$. *Ans.* $\frac{79}{15}$.

40. $y=(x+1)(x+3)$, $x=0$. *Ans.* 4.

41. $y=\frac{x}{2x+3}$, $(1,\frac{1}{5})$. *Ans.* $\frac{3}{25}$.

42. $y=\frac{x^3-3x^2-5x}{x^2+3x+2}$, $x=2$. *Ans.* $\frac{19}{72}$.

43. $y=\frac{2}{x}-\frac{3}{x^2}$, $(1,-1)$. *Ans.* 4.

44. $y=x^3+3x^2+2x$ where the curve crosses the x-axis.

For the following curves, find the points where the tangent is parallel to the x-axis.

45. $y=x^3-12x+2$.

46. $y=x^4+4x$.

47. $y=\frac{1}{3}x^3+\frac{1}{2}x^2-2x$.

48. $y=\frac{1}{2}x^4-\frac{1}{3}x^3-2$.

49. $y=\frac{1}{3}x^3-3x^2+11x+5$. *Ans.* None.

Find the rate of change of each of the following functions at the time indicated, if x is measured in feet and t in seconds.

50. $x=\frac{2\sqrt{t}}{3}-\frac{2}{3\sqrt{t}}$, $t=4$. *Ans.* $\frac{5}{24}$ ft. per sec.

51. $x=\frac{2}{3t^2-2t}$, $t=1$. *Ans.* -8 ft. per sec.

52. $x=(t^2-5t+2)(t^2+2t+3)$, $t=0$. *Ans.* -11 ft. per sec.

53. $x=(t^3-t^2)(t^2-t-1)$, $t=0$. *Ans.* 0.

54. In the proof of (1), § 16, why must n be assumed to be a positive integer?

55. Using (4′), prove formula (1), § 16, for negative integral values of n.

17. Derivative of a function of a function. Given y as a function of x, it is frequently convenient to think of y as a function of an auxiliary variable u, where u in turn is a function of x. For example, the formula of § 16 would fail entirely to find the derivative of such a function as

$$y = \sqrt{x^2 + 1};$$

it will appear presently that the difficulty may be overcome by merely putting $y = \sqrt{u}$, where $u = x^2 + 1$.

Let $$y = f(u), \text{ where } u = \phi(x).$$

Assign to x an increment Δx, and denote by Δu and Δy the corresponding changes in u and y. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

and, passing to the limit, we find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x},$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{5}$$

This very important formula is easily remembered from the fact that *in form* it is a mere identity. One of the many applications of the formula will be explained in the next article.

18. The general power formula. Suppose

$$y = u^n, \text{ where } u = \phi(x).$$

Then

$$\frac{dy}{du} = nu^{n-1},$$

and we have by (5)

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}. \tag{6}$$

An important special case of this formula is the case $n = \frac{1}{2}$:

$$\frac{d}{dx} \sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}}. \tag{6'}$$

Examples: (*a*) Find the derivative of

$$y = (3x^2 + 1)^4.$$

This function is of the form u^n, with $u = 3x^2 + 1$, $n = 4$. Hence (**6**) gives

$$y' = 4(3x^2 + 1)^3 \cdot 6x = 24x(3x^2 + 1)^3.$$

(*b*) Differentiate the function*

$$y = \sqrt{x^2 + 1}.$$

By (**6′**), we have

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}.$$

(*c*) Differentiate

$$y = (4x + 1)^2(x^2 - 2)^3.$$

By (**3**) and (**6**) we have

$$\begin{aligned} y' &= (4x + 1)^2 3(x^2 - 2)^2 2x + (x^2 - 2)^3 2(4x + 1)4 \\ &= (4x + 1)(x^2 - 2)^2 [3(4x + 1)2x + (x^2 - 2)8] \\ &= (4x + 1)(x^2 - 2)^2(32x^2 + 6x - 16). \end{aligned}$$

19. Higher derivatives. The derivative of y with respect to x is itself a function of x, and may in turn be differentiated. The derivative of the first derivative is called the *second derivative*, and is written $\frac{d^2y}{dx^2}$ (read *d* second *y* over *dx* square); further differentiations give $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, etc.

Other symbols for the higher derivatives are y'', y''', $y^{(4)}$, $\cdots$; also $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, $\cdots$.

EXERCISES

Differentiate the following functions.

1. $y = (x^2 - 1)^3$. *Ans.* $y' = 6x(x^2 - 1)^2$.

2. $y = (3x - 4)^2$. *Ans.* $y' = 6(3x - 4)$.

3. $y = \sqrt{1 - 3x}$. *Ans.* $y' = \frac{-3}{2\sqrt{1 - 3x}}$.

* Cf. the first paragraph of § 17.

4. $y = (1 - x)^{\frac{3}{2}}$. *Ans.* $y' = -\frac{3}{2}(1 - x)^{\frac{1}{2}}$.

5. $x = (y^2 - 2y)^2$. *Ans.* $\frac{dx}{dy} = 4(y^2 - 2y)(y - 1)$.

6. $z = \sqrt{a^2 - x^2}$. *Ans.* $z' = \frac{-x}{\sqrt{a^2 - x^2}}$.

7. $y = (x^3 - 1)^{\frac{4}{3}}$.

8. $u = (1 + 2v - v^2)^2$.

9. $f(x) = \sqrt{a + bx + cx^2}$.

10. $F(t) = (t^3 + 3a^2t)^2$.

11. $f(y) = (2 - 3y + 3y^2)^{-2}$.
Ans. $f'(y) = 6(1 - 2y)(2 - 3y + 3y^2)^{-3}$.

12. $y = (2ax - x^2)^{-\frac{1}{2}}$. *Ans.* $y' = (x - a)(2ax - x^2)^{-\frac{3}{2}}$.

13. $y = \frac{1}{(1 + x)^2}$. *Ans.* $y' = \frac{-2}{(1 + x)^3}$.

14. $y = \frac{1}{(2 - x)^2}$. *Ans.* $y' = \frac{2}{(2 - x)^3}$.

15. $f(x) = \frac{1}{3x - 4}$.

16. $x = \frac{2}{t^2 + 1}$.

17. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

18. $y = x(x - 1)^3$.

19. $y = x^2(1 - 3x)^2$. *Ans.* $y' = 2x(1 - 3x)(1 - 6x)$.

20. $y = x\sqrt{x + 1}$. *Ans.* $y' = \frac{3x + 2}{2\sqrt{x + 1}}$.

21. $f(x) = x^2\sqrt{a - x}$. *Ans.* $f'(x) = \frac{4ax - 5x^2}{2\sqrt{a - x}}$.

22. $\phi(y) = (y + 1)(y - 1)^2$. *Ans.* $\phi'(y) = (y - 1)(3y + 1)$.

23. $y = (1 - x)^3(1 + 2x)^2$. *Ans.* $y' = (1 - x)^2(1 + 2x)(1 - 10x)$.

24. $y = x\sqrt{a^2 - x^2}$. *Ans.* $\frac{dy}{dx} = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$.

25. $y = x\sqrt{a + bx}$. *Ans.* $\frac{dy}{dx} = \frac{2a + 3bx}{2\sqrt{a + bx}}$.

26. $r = (1 - t)\sqrt{1 + t}$. *Ans.* $\frac{dr}{dt} = \frac{-1 - 3t}{2\sqrt{1 + t}}$.

27. $y = x^2\sqrt{(a^2 - x^2)^3}$. *Ans.* $y' = x\sqrt{a^2 - x^2}(2a^2 - 5x^2)$.

28. $y = (1 - x^2)(1 + x^2)^2$. *Ans.* $y' = 2x(1 - 3x^2)(1 + x^2)$.

29. $f(x) = (1 - x^2)^4(1 - 2x)^3$.
Ans. $f'(x) = 2(1 - x^2)^3(1 - 2x)^2(11x^2 - 4x - 3)$.

30. $y = (4x + 5)^2(2x + 3)^{\frac{3}{2}}$.
Ans. $y' = (4x + 5)(2x + 3)^{\frac{1}{2}}(28x + 39)$.

31. $y = \dfrac{\sqrt{x+1}}{2x}$. *Ans.* $y' = \dfrac{-x-2}{4x^2\sqrt{x+1}}$.

32. $y = \dfrac{x^2}{(1-x)^2}$. *Ans.* $y' = \dfrac{2x}{(1-x)^3}$.

33. $\phi(x) = \dfrac{2x}{3\sqrt{2x+1}}$. *Ans.* $\phi'(x) = \dfrac{2x+2}{3(2x+1)^{\frac{3}{2}}}$.

34. $u = \dfrac{(x^2-1)^2}{x^2+1}$. *Ans.* $\dfrac{du}{dx} = \dfrac{2x(x^2-1)(x^2+3)}{(x^2+1)^2}$.

35. $y = \sqrt{\dfrac{x}{1-x}}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}\sqrt{(1-x)^3}}$.

36. $y = \dfrac{x}{(2-3x)^4}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{9x+2}{(2-3x)^5}$.

37. $y = \dfrac{x}{\sqrt{(a^2-x^2)^3}}$. *Ans.* $y' = \dfrac{2x^2+a^2}{(a^2-x^2)^{\frac{5}{2}}}$.

38. $y = \sqrt{\dfrac{1+2z}{1-2z}}$. *Ans.* $\dfrac{dy}{dz} = \dfrac{2}{(1-2z)\sqrt{1-4z^2}}$.

39. $y = \dfrac{(2x+3)^2}{(3x-1)^3}$. *Ans.* $y' = -\dfrac{(2x+3)(6x+31)}{(3x-1)^4}$.

40. $y = \left(\dfrac{x+1}{x}\right)^{\frac{1}{3}}$. *Ans.* $y' = \dfrac{-1}{3x^{\frac{4}{3}}(x+1)^{\frac{2}{3}}}$.

41. $x = \sqrt{\dfrac{y^3}{a-y}}$. *Ans.* $\dfrac{dx}{dy} = \dfrac{(3a-2y)y^{\frac{1}{2}}}{2(a-y)^{\frac{3}{2}}}$.

42. $y = \dfrac{x\sqrt{1-x}}{1+2x}$. *Ans.* $y' = \dfrac{2-3x-2x^2}{2(1+2x)^2\sqrt{1-x}}$.

43. $y = \dfrac{x(1-x)^2}{1+x}$. *Ans.* $y' = \dfrac{(1-x)(1-3x-2x^2)}{(1+x)^2}$.

44. $x = \dfrac{1}{y^2(y+3)^2}$. *Ans.* $\dfrac{dx}{dy} = -\dfrac{4y+6}{y^3(y+3)^3}$.

45. $f(x) = \dfrac{2x(1-x)^3}{3(1+x)^2}$. *Ans.* $f'(x) = \dfrac{2(1-x)^2(1-5x-2x^2)}{3(1+x)^3}$.

In the following cases, find the second derivative.

46. $y = x(x+1)^3$. *Ans.* $y'' = 6(x+1)(2x+1)$.

47. $f(x) = \sqrt{2ax-x^2}$. *Ans.* $f''(x) = \dfrac{-a^2}{(2ax-x^2)^{\frac{3}{2}}}$.

48. $x = t^2(2-3t)^3$. *Ans.* $\dfrac{d^2x}{dt^2} = 4(2-3t)(2-24t+45t^2)$.

49. $y = \sqrt{k - v}$. *Ans.* $\frac{d^2y}{dv^2} = \frac{-1}{4(k-v)^{\frac{3}{2}}}$.

50. $y = \frac{1}{\sqrt{1 - x^2}}$. *Ans.* $\frac{d^2y}{dx^2} = \frac{1 + 2x^2}{(1 - x^2)^{\frac{5}{2}}}$.

51. Find the slope of the curve $y = \frac{x^3}{(2x - 3a)^2}$ at the point $(2a, 8a)$. *Ans.* -20.

52. Find the slope of the curve $y = \sqrt{1 - 2x^3}$ at the point $x = -1$. *Ans.* $-\sqrt{3}$

53. In Exs. 3, 6, 12, 13, 14, 20, 21, 22, 24, 25, 26, 31, 32, 33, 36 above, find the second derivative of the given function.

54. Find the rate of change of the slope of the curve $y = (5x - 3)^3$ at the point $(1, 8)$. *Ans.* 300.

55. Find the rate of change of the slope of the curve $y = (1 - x^2)^3$ at the point $(-2, -27)$. *Ans.* -342.

56. If $y = (2x - 1)^4$, find y'''. *Ans.* $192(2x - 1)$.

57. If $\phi(x) = \sqrt{a + bx}$, find $\phi^{(4)}(x)$. *Ans.* $-\frac{5}{16}b^4(a + bx)^{-\frac{7}{2}}$.

58. If $y = uv$, where u and v are any functions of x, show that $y'' = uv'' + 2u'v' + u''v$. Obtain the formula for y'''.

59. If $y = \frac{u}{v}$, find y''.

60. If $f(x)$ contains the factor $(x - a)^3$, show that $f'(x)$ contains the factor $(x - a)^2$ and $f''(x)$ contains the factor $(x - a)$. Generalize this result.

20. Implicit functions. It is clear that in general an equation of any form involving x and y determines a value (or values) of y corresponding to each value of x, and therefore determines y as a function of x. Up to this point we have been concerned with functions defined explicitly by an equation of the form

$$y = f(x).$$

It may happen, however, that x and y are connected by an equation not solved for y; for example,

$$x^2 + y^2 = a^2,$$

$$x^3 + y^3 - 3axy = 0,$$

etc.

In such a case y is called an *implicit function* of x, and the relation is expressed by writing

$$F(x, y) = 0.$$

The definition becomes explicit if we solve for y; in the first example above,

$$y = \pm\sqrt{a^2 - x^2}.$$

Frequently, however, it is not desirable to change to the explicit form, even when such a change is possible.

21. Differentiation of implicit functions. To find the derivative of a function defined implicitly, we apply the following

RULE: *Differentiate each term of the equation*

$$F(x, y) = 0,$$

bearing in mind that y is a function of x.

In this connection it must be remembered that, by (**6**), the derivative of y^n with respect to x is $ny^{n-1}\dfrac{dy}{dx}$.

Examples: (*a*) Find y', if

$$x^3 + y^3 - 3axy = 0.$$

Differentiating each term in turn, we have

$$3x^2 + 3y^2y' - 3a(xy' + y) = 0,$$

whence

$$(y^2 - ax)y' + x^2 - ay = 0,$$

$$y' = -\frac{x^2 - ay}{y^2 - ax}.$$

(*b*) Find y'', if $x^2 + y^2 = a^2$.

Differentiating, we get

$$2x + 2yy' = 0,$$

whence

$$y' = -\frac{x}{y}. \tag{1}$$

A second differentiation gives

$$y'' = -\frac{y - xy'}{y^2},$$

or, after substituting the value of y' from (1),

$$y'' = -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3};$$

since $x^2 + y^2 = a^2$, this reduces to

$$y'' = -\frac{a^2}{y^3}.$$

EXERCISES

Find the derivative of y with respect to x.

1. $y^2 + ax = a^2$. *Ans.* $\frac{dy}{dx} = -\frac{a}{2y}$.

2. $x^2 + y^2 = 2ax$. *Ans.* $\frac{dy}{dx} = \frac{a - x}{y}$.

3. $x^2 - y^2 - 2x + 3y = 1$. *Ans.* $y' = \frac{2x - 2}{2y - 3}$.

4. $xy = a^2$. *Ans.* $y' = -\frac{a^2}{x^2}$.

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $y' = -\frac{b^2x}{a^2y}$.

6. $x^2 - xy + y^2 + x = 4$. *Ans.* $y' = \frac{2x - y + 1}{x - 2y}$.

7. $2x^2 - 3xy - 5y^2 - 8x + 3y + 6 = 0$.
Ans. $y' = \frac{4x - 3y - 8}{3x + 10y - 3}$.

8. $2xy + y^2 = 5$.

9. $x^3 - 2y^3 = 1$.

10. $x^2 - 3xy + 5y^2 + 6y = 7$.

11. $y^3 - 3x + 5y + 6 = 0$.

12. $x^3 + 2x^2y + y^2 - x - 2y = 0$. *Ans.* $y' = \frac{3x^2 + 4xy - 1}{2 - 2y - 2x^2}$.

13. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $y' = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$.

14. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $y' = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

15. If $x^2 - 3xt + 2t^2 - 3x = 1$, find $\frac{dx}{dt}$. *Ans.* $\frac{3x - 4t}{2x - 3t - 3}$.

16. If $2u^2 - 3uv - 5v^2 - u + v = 0$, find $\frac{dv}{du}$.

Ans. $\frac{4u - 3v - 1}{3u + 10v - 1}$.

17. If $r^3 - 2r^2z - 3z^2 = z$, find $\frac{dr}{dz}$. *Ans.* $\frac{2r^2 + 6z + 1}{3r^2 - 4rz}$.

18. If $(u + z)^3 = 2(u - z)$, find $\frac{du}{dz}$. *Ans.* $\frac{2 + 3(u + z)^2}{2 - 3(u + z)^2}$.

19. If $(u - v)(u + v)^2 = a^3$, find $\frac{dv}{du}$. *Ans.* $\frac{3u - v}{3v - u}$.

20. If $(x^2 + z^2)^3 = x$, find $\frac{dz}{dx}$. *Ans.* $\frac{1 - 6x(x^2 + z^2)^2}{6z(x^2 + z^2)^2}$.

21. If $y^2 = 4ax$, find y''. *Ans.* $y'' = -\frac{4a^2}{y^3}$.

22. If $x^2 - y^2 = a^2$, find y''. *Ans.* $y'' = -\frac{a^2}{y^3}$.

23. In Exs. 1, 2, 4, 5 above, find $\frac{d^2y}{dx^2}$.

24. If $x^3 = a^2t$, find $\frac{d^2x}{dt^2}$. *Ans.* $\frac{d^2x}{dt^2} = -\frac{2a^4}{9x^5}$.

25. If $y^3 = at^2$, find $\frac{d^2y}{dt^2}$. *Ans.* $\frac{d^2y}{dt^2} = -\frac{2a}{9y^2}$.

26. If $ay^2 = t^3$, find $\frac{d^2y}{dt^2}$. *Ans.* $\frac{d^2y}{dt^2} = \frac{3t}{4ay}$.

27. If $x^2y = a^3$, find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{d^2y}{dx^2} = \frac{6a^3}{x^4}$.

28. If $xy^2 = a^3$, find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{d^2y}{dx^2} = \frac{3y}{4x^2}$.

Find the slope of each of the following curves at the point indicated. (After differentiating, it is more convenient to substitute the given coördinates before solving for y'.)

29. $x^2 - y^2 + 2y = 1$ at (2, 3). *Ans.* 1.

30. $2x^2 + xy - y^2 = 14$ at (3, -1). *Ans.* $-\frac{11}{5}$.

31. $y^2 = \frac{x}{1 - x}$ at $(\frac{1}{2}, 1)$. *Ans.* 2.

32. $y^2 = \frac{x^2}{x - 2}$ at (3, -3). *Ans.* $\frac{1}{2}$.

33. $y^3 = \frac{a^2x^2}{2x - a}$ at (a, a). *Ans.* 0.

34. $y^2 = x(x - 1)(x + 8)$ at (4, 12). *Ans.* 4.

35. $x^3 - x^2y + 3y^2 + 2x + y = 0$ at (0, 0). *Ans.* -2.

36. $y^2 = \frac{x^2 + 3x - 2}{2x - 2}$ at (3, -2). *Ans.* $-\frac{1}{16}$.

CHAPTER IV

GEOMETRIC APPLICATIONS

22. Tangents and normals to plane curves. It is known from analytic geometry that the equation of a line of slope m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Hence, to find the tangent at any point of a plane curve we have only to find the slope of the curve $\left(i.e.\text{ the value of } \frac{dy}{dx}\right)$ at that point, and substitute for m in the above formula.

The equation of the normal — *i.e.* the line perpendicular to the tangent at the point of contact — can be found at once from that of the tangent by recalling that if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

Examples: (*a*) Find the tangent and normal to the ellipse

$$4x^2 + 9y^2 = 40$$

at the point $(1, -2)$.

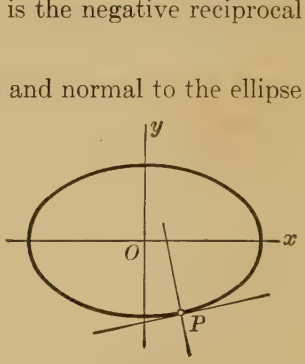


FIG. 6

We have

$$8x + 18yy' = 0,$$

or, at $(1, -2)$,

$$8 - 36y' = 0,$$

so that

$$m = \tfrac{2}{9}.$$

Therefore the equation of the tangent is

$$y + 2 = \tfrac{2}{9}(x - 1),$$

or

$$2x - 9y = 20;$$

that of the normal is

$$9x + 2y = 5.$$

(*b*) Find the equation of a tangent to the curve $y = x^3$ parallel to the line $y = 3x + 1$.

The slope of the required tangent is 3. But the slope at any point (x, y) of the curve $y = x^3$ is

$$y' = 3x^2.$$

Hence the coördinates of the point of contact are found by solving the simultaneous equations

$$3x^2 = 3,\ y = x^3.$$

This gives the points $(1, 1)$, $(-1, -1)$, and the required tangents are

$$y - 1 = 3(x - 1),\ y + 1 = 3(x + 1).$$

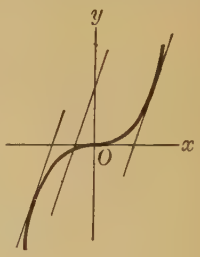


FIG. 7

EXERCISES

Find the tangent and normal to each of the following curves at the points indicated. (The answers given are the equations of the tangents.)

1. $y = x^3 - 3x^2 + 4x + 5$ at $x = 2$. *Ans.* $4x - y + 1 = 0$.
2. $y = 2x^3 - x^2 - 5$ at $x = 1$. *Ans.* $y = 4x - 8$.
3. $y = 3x^2 - 2x + 5$ at the points $y = 6$.
   Ans. $12x + 3y = 14,\ 4x - y + 2 = 0$.
4. $ay^2 = x^3$ at $x = 2a$. *Ans.* $3x \pm \sqrt{2}\,y = 2a$.
5. $y = x(1 - x)^3$ at $x = 1$.
6. $y^2 = \dfrac{x^3}{2a - x}$ at (a, a).
7. $x^2 - 2y^2 + 3x + 4y = 0$ at the points $y = 0$.
8. $y = \dfrac{a^2x}{x^2 - a^2}$ at the origin.
9. $(2x - y)^2 + x - 3y = 4$ at the points of intersection with the y-axis. *Ans.* $3x - y + 4 = 0,\ x - y - 1 = 0$.
10. $2y - 2xy = (2 - x)^2$ at $x = -1$. *Ans.* $3x + 8y = 15$.
11. $y^2 = 2x^2 - x$ at $x = 1$. *Ans.* $2y = \pm(3x - 1)$.
12. $2x^2 - 2xy + y^2 = x$ at $x = \frac{1}{2}$. *Ans.* $2x - 2y \pm 1 = 0$.
13. $y = \dfrac{(1 - 2x)^2}{(1 + 2x)^2}$ at $x = -1$. *Ans.* $y = 24x + 33$.
14. $y^2 = 4ax$ at (x_1, y_1). *Ans.* $y_1y = 2ax + 2ax_1$.
15. $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$ at (x_1, y_1). *Ans.* $\dfrac{x_1x}{a^2} \pm \dfrac{y_1y}{b^2} = 1$.

16. $12x^2 + 2y^2 + 9xy = 2$ at $(-1, 2)$. *Ans.* $6x + y + 4 = 0$.

17. $2xy = a^2$ at (x_1, y_1). *Ans.* $y_1x + x_1y = a^2$.

Find tangents to the following curves as directed.

18. To the parabola $y^2 = 3x$ parallel to the line $3x + y = 2$.
Ans. $12x + 4y + 1 = 0$.

19. To the hyperbola $xy + 8 = 0$ perpendicular to the line $x + 2y = 0$. *Ans.* $2x - y = \pm 8$.

20. To the circle $x^2 + y^2 = 10$ perpendicular to the line $3x + y = 6$.
Ans. $x - 3y = \pm 10$.

21. To the parabola $x^2 + 2x - 3y + 4 = 0$ parallel to the line $x + 2y + 1 = 0$. *Ans.* $8x + 16y = 5$.

22. To the curve $y = 3x^2 - x^3$ perpendicular to the line $x + 3y = 1$.
Ans. $y = 3x - 1$.

23. To the hyperbola $3x^2 - 2y^2 = 10$ having the slope -3.
Ans. $y + 3x = \pm 5$.

24. To the curve $x^4 + y = 2$ perpendicular to the line $2x + y = 3$.
Ans. $8x - 16y + 35 = 0$.

25. To the curve $y = \frac{1}{4}x^4 - x^3 + 5x$ making an angle of 45° with Ox. *Ans.* $4x - 4y = 11$, $x - y + 4 = 0$.

26. To the curve $y = 1 - 2x - 2x^2$ parallel to Ox. *Ans.* $y = \frac{3}{2}$.

27. To the curve $y = \frac{1}{4}x^4 - \frac{3}{2}x^2 - 4x$ perpendicular to the line $x - 2y = 0$. *Ans.* $8x + 4y = 3$, $2x + y + 6 = 0$.

28. Find the normal to the parabola $y = 1 - x^2$ having the slope 2.
Ans. $32x - 16y + 7 = 0$.

29. Find a normal to the curve $a^2y = x^3$ having the slope $-\frac{1}{6}$.
Ans. $x + 6y = \pm 13\sqrt{2}\,a$.

30. Find a normal to the parabola $y^2 = x$ perpendicular to the line $2x - 3y = 3$. *Ans.* $48x + 32y = 51$.

31. Find the tangents to the curve $y = x^4 - 4x^3 - 18x^2 + 3$ at the points where $y'' = 0$. *Ans.* $y = 20x + 10$, $108x + y = 138$.

32. Find the tangents of slope 2 to the curve $4y = x^4 - 26x^2 + 56x + 4$. *Ans.* $8x - 4y + 27 = 0$, $8x - 4y = 5$, $2x - y = 87$.

33. Show that the area of the triangle formed by the coördinate axes and the tangent to the hyperbola $2xy = a^2$ is constant.

34. Show that the sum of the intercepts on the axes of the tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is constant.

35. Show that the portion of the tangent to the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the axes is constant.

36. Determine a, b, c so that the parabola $y = ax^2 + bx + c$ shall touch the line $y = 3x - 2$ at $(2, 4)$ and shall also pass through $(0, 0)$. Check the result, and draw the curve. *Ans.* $y = \frac{1}{2}x^2 + x$.

37. Make the parabola $y = ax^2 + bx + c$ pass through $(-3, 1)$ and touch the line $2x + 3y = 4$ at $(-1, 2)$. Check the answer, and draw the curve.

38. Make the parabola $y^2 + ax + by + c = 0$ pass through $(4, 1)$ and touch the line $x - y = 5$ at $(2, -3)$.

39. Find the equation of a circle passing through the points $(1, 3)$, $(2, 0)$ and touching the line $x + 3y = 10$ at the former point.
Ans. $4x^2 + 4y^2 - 3x - 9y - 10 = 0$.

40. Find the equation of an ellipse with axes lying in the coördinate axes and touching the line $3x + 5y = 11$ at $(2, 1)$.
Ans. $3x^2 + 10y^2 = 22$.

23. Graphs. In constructing the graph of a given function, we have relied hitherto on the elementary method of plotting by separate points. This method, although useful in giving an accurate detail of some portion of the curve in which we may be specially interested, fails to exhibit the general properties of the function. In the next few articles we shall develop methods by which, in many cases, the general appearance of the curve may be very quickly discovered. These methods will rest chiefly on a study of the first and second derivatives of the function.

For simplicity, we shall in this chapter confine our attention to the graphs of polynomials. A fuller treatment of the subject of curve tracing will be given in Chapter XI.

24. Preliminary steps. In tracing a curve, it is well to begin by finding the points of intersection with the axes, provided this can be done without great difficulty.

The behavior of a function for large values of x, both positive and negative, can usually be determined at once by inspection (cf. § 11). This question should always be investigated, since the result is of great value in interpreting the meaning of other data.

25. Increasing and decreasing functions. We proceed to show how the calculus may be used to obtain important information concerning the graph. While the results will at present be employed in the study of polynomials only, they apply in general.

Given

$$y = f(x),$$

we note first that, as x increases, *the curve rises if the slope is positive,* as on the arc AB (Fig. 8); *it falls if the slope is negative,* as along BD:

If $y' > 0$, y increases;

If $y' < 0$, y decreases.

Of course this also appears at once from the fact that y' is the rate of change of y.

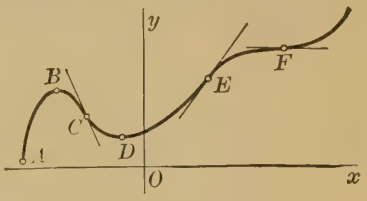


FIG. 8

26. Maxima and minima. At a point such as B (Fig. 8), where the function is algebraically greater than at any neighboring point, the function is said to have a *maximum value,* and the point is called a *maximum point.* Similarly, at D the function has a *minimum value.* It is evident that *at such points the tangent is parallel to Ox; i.e.*

$$y' = 0.$$

But the vanishing of the derivative does not mean that the function is necessarily a maximum or a minimum; the tangent is parallel to Ox at F, yet the function is neither a maximum nor a minimum there. It appears from the figure that the test is as follows:

At a point where $y' = 0$, if y' changes from positive to negative (as x increases), y is a maximum; if y' changes from negative to positive, y is a minimum; if y' does not change sign, y is neither a maximum nor a minimum.

The points at which $y' = 0$ are called *critical points,* and the corresponding values of x are the *critical values* of x.

27. Concavity. The second derivative is the rate of change of the first derivative. It follows that when y'' is positive, y' is increasing; as x increases the tangent turns in counter-clockwise sense and the curve is *concave upward.* When y'' is negative, y' decreases; the curve is *concave downward.*

At a maximum point the curve is concave downward, and hence y'', if it is not 0, must be negative. At a minimum y'', if not 0, must be positive. If the second derivative is easily obtained and if it does not happen to be 0 at the critical point in question, it is usually more convenient to determine whether we have a maximum or a minimum by finding the sign of y''; but the test of § 26 has the advantage of being perfectly general. However, in practice it is seldom necessary to apply either of these tests, since the nature of the various critical points can almost always be determined very easily by inspection.

Example: $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2.$

When x is sufficiently large, the term in x^3 will be numerically greater than the other terms combined, so that y will have the same sign as x^3. Thus, when x is large and negative, y is large and negative; when x is large and positive, y is large and positive.

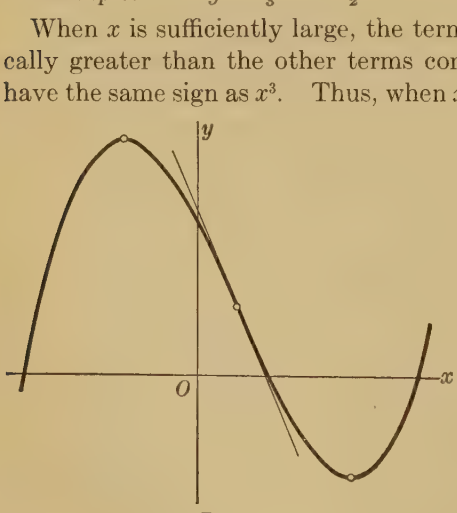


FIG. 9

The first derivative is

$$y' = x^2 - x - 2;$$

setting

$$x^2 - x - 2 = 0,$$

we get the critical values $x = -1$ or 2, and the critical points are $(-1, \frac{19}{6})$, $(2, -\frac{4}{3})$. Since y is large and negative when x is large and negative, the curve comes up through the third quadrant,

rises to a maximum at $(-1, \frac{19}{6})$, crosses Oy at $(0, 2)$, falls to a minimum at $(2, -\frac{4}{3})$, and then rises indefinitely (since when x is large and positive y is large and positive). The conclusion that $(-1, \frac{19}{6})$ is a maximum and $(2, -\frac{4}{3})$ a minimum may be verified by either of the tests mentioned above.

The curve is drawn in the figure.

EXERCISES

In the following cases, determine the behavior of the function for large values of the argument, locate the critical points, and distinguish between maxima and minima.

1. $y = 2x^2 - 4x + 3$. *Ans.* $(1, 1)$, minimum.

2. $y = 1 - x - x^2$. *Ans.* $(-\frac{1}{2}, \frac{5}{4})$, maximum.

3. $y = 12x - x^2$.

4. $y = -\frac{1}{4}(x - 2)^2$.

5. $y = \frac{x^2}{4} - \frac{3x}{2} + 1$.

6. $y = \frac{x^2}{3} - \frac{x}{4} + \frac{1}{2}$.

7. $y = x^3 - 3x$.

8. $y = 1 + x - \frac{1}{3}x^3$.

9. $y = 2 + 8x - \frac{2}{3}x^3$.

10. $a^2y = x(x - a)^2$.

11. $y = x^2(3 - x)$.

12. $3y = x(2x + 1)^2$.

13. $y = \frac{x(3 - 2x)^2}{4}$.

14. $y = \frac{x^3}{3} - \frac{3x^2}{2} - 4x + 1$.

15. $y = -x^3 - 9x^2 - 24x$.

16. $y = x^3 - x^2 - x + 1$.

17. $y = x^3 + x^2 + 4x - 2$. *Ans.* No critical points.

18. $y = 1 - 2x - 3x^2 - 4x^3$. *Ans.* No critical points.

19. $y = x^2(x - 1)^2$. *Ans.* $(\frac{1}{2}, \frac{1}{16})$, maximum; $(0, 0)$, $(1, 0)$, minima.

20. $y = x^2(8 - x^2)$. *Ans.* $(0, 0)$, minimum; $(\pm 2, 16)$, maxima.

21. $a^3y = (x^2 - a^2)^2$. *Ans.* $(0, a)$, maximum; $(\pm a, 0)$, minima.

22. $y = (x^2 + 1)(x + 1)^2$. *Ans.* $(-1, 0)$, minimum.

23. $y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + 1$. *Ans.* $(1, \frac{5}{12})$, minimum.

24. $y = 1 - 2x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4$.
Ans. $(1, -\frac{1}{12})$, minimum; $(-1, \frac{31}{12})$, $(2, \frac{1}{3})$, maxima.

25. $a^2y = x^3$. *Ans.* No maxima or minima.

26. $y = x(x - 1)^3$. *Ans.* $(\frac{1}{4}, -\frac{27}{256})$, minimum.

27. $y = \frac{1}{5}x^5 - \frac{1}{3}x^3$.
Ans. $(-1, \frac{2}{15})$, maximum; $(1, -\frac{2}{15})$, minimum.

28. $y = 2x^6 - 3x^4$. *Ans.* $(\pm 1, -1)$, minima; $(0, 0)$, maximum.

28. Points of inflection. A *point of inflection* is a point at which the curve changes from concave upward to concave downward, or vice versa (the points C, E, F in Fig. 8, p. 37).

At a point of inflection the tangent reverses the sense in which it turns, which means that y' changes from an increasing to a decreasing function, or vice versa. Hence at such a point y'' changes sign, and, if it is continuous, must vanish. Conversely, *a point at which y'' vanishes is a point of inflection, provided y'' changes sign at that point.*

Since y'' — *i.e.* the rate of change of the slope — is 0 at a point of inflection, the tangent is sometimes said to be *stationary for an instant* at such a point, and in the neighborhood of the point it turns very slowly. Hence the inflectional tangent agrees more closely with the curve near its point of contact than does an ordinary tangent; it is therefore especially useful in tracing the curve to *draw the tangent at each point of inflection.*

A point at which y'' vanishes without changing sign is not a point of inflection; the result merely means that near that point the tangent turns very slowly.

As noted in §§ 26–27, a point where $y' = 0$ is a maximum or a minimum, provided $y'' \neq 0$. If y' and y'' both equal 0, the point is in general a point of inflection with a horizontal tangent (the point F in Fig. 8); but if y'' vanishes without changing sign, the point is a maximum or minimum.

29. Summary. The theory of §§ 24–28 may now be summarized in the form of a definite sequence of steps, as follows:

(1) *Find the points of intersection with the axes.*

(2) *Determine the behavior of y for large values of x.*

(3) *Locate the points where $y' = 0$, and determine the maxima and minima.*

(4) *Locate the points where $y'' = 0$ (points of inflection, in general), and draw the tangent at each of those points.*

Examples: (*a*) Trace the curve

$$y = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 2x + 2.$$

(1) When $x = 0$, $y = 2$. The equation $y = 0$ has no rational roots; the determination of intersections with Ox will therefore be omitted.

(2) When x is large and negative, y is large and negative; when x is large and positive, y is large and positive.

(3) The first derivative is

$$y' = x^2 - x - 2.$$

It has been found in § 27 that $(-1, \frac{19}{6})$ is a maximum and $(2, -\frac{4}{3})$ a minimum.

(4) The second derivative is

$$y'' = 2x - 1.$$

Equating this to 0, we get $x = \frac{1}{2}$; the point $(\frac{1}{2}, \frac{11}{12})$ is a point of inflection, the slope at that point being $-\frac{9}{4}$.

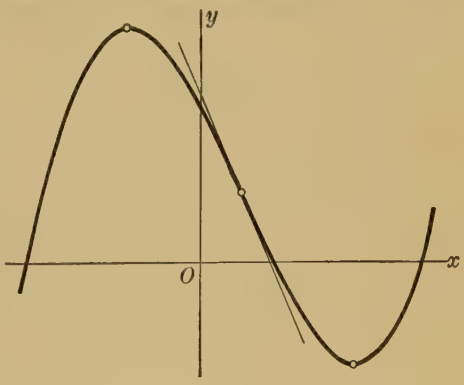


FIG. 10

The curve appears in Fig. 10.

(*b*) Trace the curve $y = x(x-1)^3$.

(1) When $x = 0$, $y = 0$; when $y = 0$, $x = 0$ or 1.

(2) When x is large and either positive or negative, y is large and positive.

(3) We find

$$y' = (x-1)^3 + 3x(x-1)^2$$
$$= (x-1)^2(4x-1).$$

Thus the critical points are $(\frac{1}{4}, -\frac{27}{256})$, $(1, 0)$. The result of step (2) indicates that the former point is a minimum, the latter a point of inflection with horizontal tangent; this conclusion will be verified in the next step.

(4) The second derivative is

$$y'' = 2(x-1)(4x-1) + 4(x-1)^2$$
$$= 6(x-1)(2x-1).$$

The points of inflection are $(\frac{1}{2}, -\frac{1}{16})$, with slope $\frac{1}{4}$, and $(1, 0)$, with slope 0.

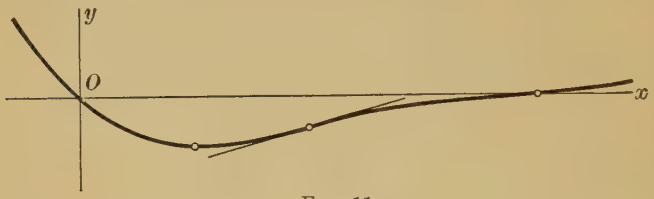


FIG. 11

The curve is shown in Fig. 11.

EXERCISES

Trace the following curves neatly, choosing a suitable scale in each case.

1. $y = \frac{1}{3}x^3 - 2x^2 + 3x - 2$.
2. $y = \frac{1}{12}x^3 - \frac{1}{4}x^2 - 2x + 1$.
3. $y = (x-2)^2(x-3)$.
4. $y = 4 - 3x^2 - x^3$.
5. $a^2y = x^2(a-x)$.
6. $y = 4 - 8x - 3x^2 - \frac{1}{3}x^3$.
7. $y = 2 + 3x + \frac{x^2}{4} - \frac{x^3}{6}$.
8. $y = \frac{(1+x)(3-2x)^2}{3}$.
9. $y = 1 - 6x - 9x^2 - 4x^3$.
10. $y = x^3 - 3x^2 + 4x - 3$.
11. $y = \frac{2}{3}x^3 + \frac{3}{2}x$.
12. $y = \frac{2}{3}x^3 - \frac{3}{2}x$.
13. $y = 2x^3 - 3$.
14. $y = (1-2x)^3$.
15. $y = 3x + \frac{3}{2}x^2 - \frac{1}{3}x^3$.
16. $y = 3x - 6x^2 + 4x^3$.
17. $y = x^3 + 3x^2 + 3x$.
18. $y = x^3 + \frac{3}{2}x^2 + 3x$.
19. $y = \frac{1}{4}x^4 - \frac{1}{2}x^2$.
20. $y = x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$.
21. $y = 3x^4 - 4x^3$.
22. $y = \frac{1}{12}(x^4 + 6x^2 - 7)$.
23. $y = (x^2+1)(x+1)^2$.
24. $a^3y = (x^2-a^2)^2$.
25. $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 + 1$.
26. $15y = 3x^5 - 20x^3$.
27. $y = (x-2)(x+1)^3$.
28. $y = \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2$.
29. $y = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + 1$.
30. $y = x(x^2-1)^2$.
31. $y = x^4 - 8x^3 + 24x^2$.
32. $12y = 32x - x^4$.
33. $y = \frac{1}{5}x^5 - \frac{1}{4}x^4$.
34. $y = \frac{1}{5}x^5 - \frac{2}{3}x^3 + 2x$.

30. Applications of the theory of maxima and minima. It was shown in § 26 that, at a point where its first derivative vanishes, a function assumes a maximum or a minimum value (provided the derivative changes sign at that point). This result finds application in a great variety of problems, some of which will now be considered.

When the derivative is equated to 0, it may happen, of course, that several critical values are obtained. In practice, the value that gives the desired maximum or minimum can usually be selected at once by inspection.

Example: A box is to be made of a piece of cardboard 4 in. square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Let x be the length of the side of each of the squares cut out. Then the volume of the box is

$$V = x(4 - 2x)^2. \tag{1}$$

The derivative is

$$V' = (4 - 2x)^2 - 4x(4 - 2x)$$
$$= (4 - 2x)(4 - 6x).$$

Setting

$$V' = 0,$$

we find

$$x = 2 \text{ or } \tfrac{2}{3}.$$

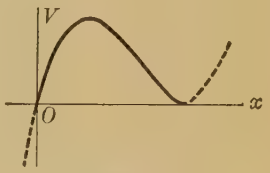


FIG. 12

Since the volume is 0 when $x = 0$ and again when $x = 2$, it must reach a maximum at some intermediate point; it therefore follows without the application of further tests that the critical value $x = \frac{2}{3}$ gives the required maximum volume:

$$V_{max.} = \tfrac{2}{3}(4 - \tfrac{4}{3})^2 = \tfrac{128}{27} \text{ cu. in.}$$

The graph of equation (1) is shown in the figure, the V-scale being one fourth as large as the x-scale. Since by the conditions of the problem x is restricted to values between 0 and 2, the dotted portions of the curve have no meaning in this case.

31. A variation of the method. If, as frequently happens, the function whose maximum or minimum is to be found can be most simply expressed in terms of two variables, a relation between these two variables must be found from the conditions of the problem. By means of this relation one of the variables can be eliminated, after which the maxima and minima can be found in the usual way. However, it is often more convenient not to eliminate, but to proceed as in the following

Example: Find the shape of the largest rectangle that can be inscribed in a given circle.

Take the coördinate axes parallel to the sides of the rectangle. The area of the rectangle is

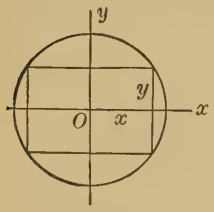


FIG. 13

$$(1) \qquad A = 4xy.$$

Further,

$$(2) \qquad x^2 + y^2 = a^2,$$

where a is the radius of the circle. Differentiating (1) with respect to x and setting the derivative equal to 0, we have, since y is a function of x,

$$A' = 4(xy' + y) = 0,$$

or

$$y' = -\frac{y}{x}.$$

Differentiating equation (2), we get

$$2x + 2yy' = 0,$$

or

$$y' = -\frac{x}{y}.$$

Equating values of y', we find

$$-\frac{y}{x} = -\frac{x}{y},$$

whence

$$y = x:$$

the maximum rectangle is a square.

EXERCISES

1. Find two numbers whose sum is unity, and whose product is a maximum. *Ans.* $\frac{1}{2}, \frac{1}{2}$.

2. What should be the shape of a rectangular field of given area, if it is to be inclosed by the least amount of fencing? *Ans.* A square.

3. A rectangular field is to be fenced off adjacent to a highway. If the fence along the highway is to cost 50% more per rod than on the other sides, find the most economical proportions. *Ans.* $x = \frac{4}{5}y$.

4. Find the rectangle of maximum perimeter inscribed in a given circle. *Ans.* A square.

5. Find the most economical proportions for a quart can.
Ans. Diameter = height.

6. Find the most economical proportions for a cylindrical cup.
Ans. Radius = height.

7. Find the most economical proportions for a box with an open top and a square base. *Ans.* Side of base = 2 × altitude.

8. Find the most economical proportions for a covered box whose base is a rectangle with one side three times the other.
Ans. Altitude = $\frac{3}{2}$ × shorter side.

9. Solve Ex. 8 if the box has an open top.
Ans. Altitude = $\frac{3}{4}$ × shorter side.

10. Find the volume of the largest box that can be made by cutting equal squares out of the corners of a piece of cardboard 6 × 16 in. and turning up the sides. *Ans.* $\frac{1600}{27}$ cu. in.

11. Solve Ex. 10 if the cardboard is 15 × 24 in. *Ans.* 486 cu. in.

12. A rectangular field is to be inclosed, and divided into three lots by parallels to one of the sides. What should be the shape of the field if the amount of fencing is to be a minimum?
Ans. Length = 2 × breadth.

13. A cylindrical glass jar has a metal top. If the metal is half as expensive as the glass, per unit area, find the most economical proportions. *Ans.* Radius = $\frac{2}{3}$ × height.

14. A triangular lot is bounded by two streets and an alley, the streets intersecting at right angles. Find the dimensions of the largest rectangular building that can be placed on the lot, facing one of the streets. *Ans.* $x = \frac{1}{2}a$.

15. Find the proportions of the largest right circular cylinder that can be inscribed in a given sphere. *Ans.* Diameter = $\sqrt{2}$ × height.

16. In Ex. 15, find the shape of the cylinder if its convex surface is to be a maximum. *Ans.* Diameter = height.

17. The base of a box is a square. The top and front are made of oak, the remainder of pine. If oak is twice as expensive as pine, find the most economical proportions. *Ans.* Side of base $= \frac{5}{6} \times$ height.

18. A cylindrical tin boiler has a copper bottom. If sheet copper is four times as expensive as tin, per unit area, find the most economical proportions. *Ans.* Height $= 4 \times$ radius.

19. Solve Ex. 18 if the boiler is to have a tin cover.

Ans. Height $= 5 \times$ radius.

20. The strength of a rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of given diameter.

Ans. Depth $= \sqrt{2} \times$ breadth.

21. The stiffness of a rectangular beam is proportional to the breadth and the cube of the depth. Find the shape of the stiffest beam that can be cut from a log of given diameter. *Ans.* Depth $= \sqrt{3} \times$ breadth.

22. Find the most economical proportions for a cylindrical cup, if the bottom is three times as thick as the sides.

Ans. Diameter $= \frac{2}{3} \times$ height.

23. The perimeter of an isosceles triangle is 10 inches. Find the maximum area. *Ans.* $A = \frac{25}{9}\sqrt{3}$ sq. in.

24. Find the largest right circular cone that can be constructed having a given slant height. *Ans.* Radius $= \sqrt{2} \times$ height.

25. Find the most economical proportions for a conical tent of given capacity. *Ans.* $h = \sqrt{2}\, r$.

26. A Norman window consists of a rectangle surmounted by a semicircle. What shape gives the most light for a given perimeter?

Ans. Breadth = height.

FIG. 14

27. A gutter is to be made of a strip of tin 12 in. wide, the cross section having the form shown in the figure. What depth gives a maximum carrying capacity? *Ans.* $2\sqrt{3}$ in.

28. Find the most economical proportions for an A-tent of given volume whose sides slope at 45° to the horizontal.

Ans. Length $= \sqrt{2} \times$ height.

29. Find the dimensions of the largest right circular cylinder that can be inscribed in a given right circular cone. *Ans.* Altitude $= \frac{1}{3} h$.

30. Solve Ex. 29 if the convex surface of the cylinder is to be a maximum. *Ans.* Radius $= \frac{1}{2} a$.

31. Find the shape of the largest isosceles triangle inscribed in a given circle. *Ans.* Equilateral.

32. Find the right circular cone of maximum volume inscribed in a given sphere. *Ans.* Altitude $= \frac{4}{3}\,a$.

33. A wall tent 12×16 ft. is to contain a given volume. Find the most economical proportions. *Ans.* Height above eaves $= 2\sqrt{3}$ ft.

34. An oil can is made in the shape of a cylinder surmounted by a cone. If the radius of the cone is three fourths of its height, find the most economical proportions.

Ans. Altitude of cylinder = altitude of cone.

35. A cupboard 5 ft. high and having shelves 1 ft. apart is to be made from a given amount of material. If a front, but no back, is required, what shape gives the greatest amount of shelf room?

Ans. Width = twice depth.

36. A silo is made in the form of a cylinder, with a hemispherical roof; there is a floor of the same thickness as the wall and roof. Find the most economical shape. *Ans.* Diameter = total height.

37. Find the right circular cone of minimum volume circumscribed about a given sphere. *Ans.* Altitude $= 4\,a$.

38. A man in a rowboat 6 miles from shore desires to reach a point on the shore at a distance of 10 miles from his present position. If he can walk 4 miles per hour and row 3 miles per hour, where should he land in order to reach his destination in the shortest possible time?

Ans. 1.2 miles from his destination.

32. Derived curves. The curves $y = f'(x)$, $y = f''(x)$, $y = f'''(x)$, $\cdots$ are called the *first, second, third, $\cdots$ derived curves*, corresponding to the curve $y = f(x)$.

The relations between the given function and its first and second derivatives, which have been formulated analytically in §§ 25–28, are well brought out graphically by drawing the original curve and its first and second derived curves. This is shown in Fig. 15 (p. 48) for the curve

$$y = \tfrac{1}{3}x^3 - x^2.$$

The work should be arranged with the several axes of ordinates lying in the same vertical line.

It often happens in practical work that a function is defined in such a way — for instance by experimental data — that no mathematical expression for it is known. If then we wish

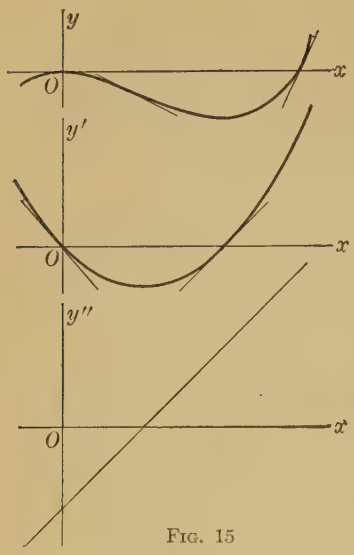


FIG. 15

to examine the behavior of one of the derivatives, as is frequently the case, we plot the graph of the original function from the given data, and construct the required derived curve graphically. The process is obvious. We can measure the slope at any point of the original curve; the number thus obtained is the ordinate of the corresponding point on the first derived curve. In this way as many points as desired may be plotted on the first derived curve and a smooth curve drawn through them, after which the second derived curve may be constructed in a similar way; etc.

EXERCISES

1. What can be said of the first derived curve

(*a*) in an interval where the original curve is $\left\{\begin{matrix}\text{rising}\\\text{falling}\end{matrix}\right\}$?

(*b*) in an interval where the original curve is concave $\left\{\begin{matrix}\text{upward}\\\text{downward}\end{matrix}\right\}$?

(*c*) at a point where the original curve has a $\left\{\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}\right\}$?

(*d*) at a point where the original curve has a point of inflection?

2. What can be said of the second derived curve

(*a*) in an interval where the original curve is concave $\left\{\begin{matrix}\text{upward}\\\text{downward}\end{matrix}\right\}$?

(*b*) at a point where the original curve has a point of inflection?

(*c*) in an interval where the first derived curve is $\left\{\begin{matrix}\text{rising}\\\text{falling}\end{matrix}\right\}$?

(*d*) at a point where the first derived curve has a $\left\{\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}\right\}$?

3. What can be said of the original curve at a point where the second derived curve touches the x-axis without crossing it?

Plot the following curves; then construct the first and second derived curves by the method described in the last paragraph of § 32.

4. $y = x^2$.

5. $y = 2x^2 - 4x + 2$.

6. $y = 1 - x - x^2$.

7. $y = 1 - x - \frac{1}{6}x^3$.

8. $y = (1 - x)^3$.

9. $y = \frac{1}{4}x^4 - x^3 + x^2$.

10. Plot the curve $y = \sin x$ and its first and second derived curves, using radian measure. What well-known curve does the first derived curve resemble? How does the second derived curve compare with the original?

11. Plot the curve $y = \log_{10} x$ and its first derived curve. What familiar curve does the latter resemble?

12. Draw a smooth curve, on a large scale, through the points

x	-4	-2	0	2	4	6	8	10	12	14	16	18	20	22	24
y	0	-1	$-\frac{3}{2}$	0	6	10	3	0	-1	-1	$-\frac{1}{2}$	1	9	20	35

and construct the first and second derived curves.

CHAPTER V

TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

I. Trigonometric Functions

33. Elementary properties. The student is already familiar with the elementary properties of the trigonometric functions. They are one-valued and continuous for all values of the argument x, except that the tangent and secant become infinite when $x = \pm (2n + 1)\frac{\pi}{2}$, the cotangent and cosecant become infinite when $x = \pm n\pi$, where n is 0 or a positive integer. The sine and cosine, and their reciprocals the cosecant and secant, are periodic with the period 2π; the tangent and cotangent are periodic with the period π.

34. Differentiation of sin x. The derivative of $\sin x$ may be obtained directly from the definition of the derivative (§ 12). We have

$$\begin{aligned} y &= \sin x, \\ y + \Delta y &= \sin (x + \Delta x), \\ \Delta y &= \sin (x + \Delta x) - \sin x, \\ \frac{\Delta y}{\Delta x} &= \frac{\sin (x + \Delta x) - \sin x}{\Delta x}. \end{aligned}$$

Expanding $\sin (x + \Delta x)$ by the addition formula of trigonometry, we get

$$\frac{\Delta y}{\Delta x} = \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}.$$

By trigonometry,

$$\cos \Delta x = 1 - 2 \sin^2 \tfrac{1}{2} \Delta x,$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{\cos x \sin \Delta x - 2 \sin x \sin^2 \frac{1}{2}\Delta x}{\Delta x}$$

$$(1) \qquad = \cos x \cdot \frac{\sin \Delta x}{\Delta x} - \sin x \cdot \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \cdot \sin \tfrac{1}{2}\Delta x.$$

It will be shown in the next article that

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

Assuming this result for the moment, we see that

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1, \quad \lim_{\Delta x \to 0} \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} = 1.$$

Hence, passing to the limit in equation (1), we find

$$\frac{dy}{dx} = \frac{d}{dx} \sin x = \cos x.$$

If u is any function of x, it follows from formula (**5**) of § 17 that

$$\frac{d}{dx} \sin u = \frac{d}{du} \sin u \cdot \frac{du}{dx},$$

or

$$(7) \qquad \frac{d}{dx} \sin u = \cos u \frac{du}{dx}.$$

Examples: (*a*) Differentiate $\sin 5x^2$.

By (**7**), with $u = 5x^2$,

$$\frac{d}{dx} \sin 5x^2 = 10x \cos 5x^2.$$

(*b*) Differentiate $\sin^3 2x$.

By formula (**6**) of § 18, together with (**7**),

$$\frac{d}{dx} \sin^3 2x = 3 \sin^2 2x \cdot \cos 2x \cdot 2$$
$$= 6 \sin^2 2x \cos 2x.$$

35. Limit of sin α/α as α approaches 0. In the differentiation of $\sin x$ we had to make use of the fact that

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

This result may be obtained as follows.

Let P, Q be two points on a circle such that the chord PQ subtends an angle $2\alpha < \pi$. As α approaches 0, the ratio of the chord to the arc approaches unity:

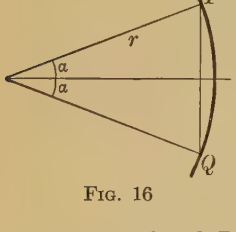


FIG. 16

$$\lim_{\alpha \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

But

$$\text{chord } PQ = 2r \sin \alpha,$$

and, *if α is measured in radians*,

$$\text{arc } PQ = 2r\alpha.$$

Hence

$$\lim_{\alpha \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = \lim_{\alpha \to 0} \frac{2r \sin \alpha}{2r\alpha} = \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

When α is in degrees, the length of the arc is

$$\text{arc } PQ = 2r \cdot \frac{\pi}{180}\alpha,$$

and the formula for the derivative of $\sin x$ becomes

$$\frac{d}{dx} \sin x = \frac{\pi}{180} \cos x.$$

In order to avoid the troublesome numerical factor $\frac{\pi}{180}$, *radian measure of angles is always employed in the calculus* unless the contrary is stated.

36. Differentiation of $\cos x$, $\tan x$, etc. The derivatives of the other trigonometric functions can also be obtained directly from the definition of the derivative, but they are more easily found from (**7**).

To differentiate $\cos x$, we write

$$\cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$$

$$= -\sin x.$$

If u is any function of x, we find by formula (**5**) of § 17,

$$\textbf{(8)} \qquad \frac{d}{dx} \cos u = -\sin u \frac{du}{dx}.$$

The remaining trigonometric functions may be differentiated by expressing them in terms of the sine and cosine. The results are as follows:

$$\frac{d}{dx}\tan x = \sec^2 x,$$

$$\frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x \tan x,$$

$$\frac{d}{dx}\csc x = -\csc x \cot x.$$

If u is any function of x, we find

(9)
$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx},$$

$$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx},$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx},$$

$$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}.$$

EXERCISES

Perform the indicated differentiations.

1. $\frac{d}{dx}\sin 2x$. *Ans.* $2\cos 2x$.

2. $\frac{d}{d\theta}\cos\frac{\theta}{2}$. *Ans.* $-\frac{1}{2}\sin\frac{\theta}{2}$.

3. $\frac{d}{d\theta}\cos\left(\frac{\pi}{4}-\theta\right)$. *Ans.* $\sin\left(\frac{\pi}{4}-\theta\right)$.

4. $\frac{d}{dx}\sin\frac{1-x}{3}$. *Ans.* $-\frac{1}{3}\cos\frac{1-x}{3}$.

5. $\frac{d}{dx}\sin\frac{3x}{2}$.

6. $\frac{d}{dy}3\cos\frac{2y}{3}$.

7. $\frac{d}{dx}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

8. $\frac{d}{dx}\frac{\tan 2x}{2}$.

9. $\frac{d}{d\phi}(\sec 3\phi + \tan 3\phi)$.

10. $\frac{d}{d\phi}\csc(1-\phi)$.

11. $\frac{d}{d\theta}3\cot\frac{\theta}{3}$.

12. $\frac{d}{dx}\sec\left(\frac{\pi}{4}+2x\right)$.

13. $\frac{d}{dx}\cos x^2$.

14. $\frac{d}{dx}\tan(1-x^2)$.

15. $\frac{d}{dt}\sin\frac{3t-1}{4}$.

16. $\frac{d}{dy}\tan\left(\frac{\pi}{4}+y^2\right)$.

Differentiate the following functions.

17. $y = x\sin x$. *Ans.* $y' = x\cos x + \sin x$.

18. $y = 2t^2\cos 2t$. *Ans.* $\frac{dy}{dt} = 4t(\cos 2t - t\sin 2t)$.

19. $x = (1-t)\sin 2t$. *Ans.* $\frac{dx}{dt} = 2(1-t)\cos 2t - \sin 2t$.

20. $y = x\tan 2x$. *Ans.* $\frac{dy}{dx} = \tan 2x + 2x\sec^2 2x$.

21. $y = \sin\theta\cos\theta$. *Ans.* $\frac{dy}{d\theta} = \cos^2\theta - \sin^2\theta$.

22. $x = \sin^2 2\theta$. *Ans.* $\frac{dx}{d\theta} = 4\sin 2\theta\cos 2\theta$.

23. $y = \cos^4\theta$. *Ans.* $\frac{dy}{d\theta} = -4\cos^3\theta\sin\theta$.

24. $y = x\sec x$. *Ans.* $y' = \sec x(1 + x\tan x)$.

25. $r = \tan^2\theta$.

26. $r = \cos^2\frac{1}{2}\theta$.

27. $x = \cot^3\frac{z}{3}$. *Ans.* $\frac{dx}{dz} = -\cot^2\frac{z}{3}\csc^2\frac{z}{3}$.

28. $y = (1-\cos\theta)^3$. *Ans.* $\frac{dy}{d\theta} = 3\sin\theta(1-\cos\theta)^2$.

29. $y = \tan^3 2x$.

30. $y = \sin\theta\sin 3\theta$.

31. $y = \sin x\cos 2x$.

32. $y = \cos^3 x - \sin^3 x$.

33. $y = \sqrt{1-\cos x}$.

34. $y = \sqrt{1-\sin 2x}$.

35. $y = \csc^2\left(\frac{\pi}{4} - 2\phi\right)$.

36. $z = x\sin^2 3x$.

37. $f(x) = \sin^2 x\cos^2 x$. *Ans.* $f'(x) = 2\sin x\cos x(\cos^2 x - \sin^2 x)$.

38. $F(x) = (1-\sin 2x)^2$. *Ans.* $F'(x) = -4\cos 2x(1-\sin 2x)$.

39. $u = \frac{\sin 2v}{2v}$.

40. $r = \frac{\sin\theta}{\cos^2\theta}$.

41. $y = x\tan^2\frac{1}{2}x$.

42. $x = \sec 2t\tan 2t$.

43. $y = (\sin^2 2\theta - 1)^2$.

44. $u = 2(\theta - \sin\theta)^2$.

45. If $y = \sin x$, determine y'', y''', $y^{(4)}$.

46. If $x = \cos kt$, find $\frac{d^2x}{dt^2}$.

47. If $x = A\cos kt + B\sin kt$, show that $\frac{d^2x}{dt^2} = -k^2x$.

48. From the formula $\sin 2x = 2 \sin x \cos x$, deduce by differentiation the formula for $\cos 2x$.

49. From the trigonometric formula for $\cos(x + \alpha)$, deduce by differentiation the formula for $\sin(x + \alpha)$.

50. If $x = t \cos kt$, find $\frac{d^2x}{dt^2}$. *Ans.* $-2k \sin kt - k^2 t \cos kt$.

37. Graphs of trigonometric functions. To plot the graph of the function

$$y = \sin x,$$

we may proceed as in § 29. On account of the periodicity of the sine function, it will be sufficient to determine the appearance of the curve in the interval from $x = 0$ to $x = 2\pi$; the remainder of the curve must consist of repetitions of this portion.

(1) When $x = 0$, $y = 0$; when $y = 0$, $x = 0, \pi, 2\pi$.

(2) Evidently the behavior of the function for large values of x need not be considered at this point.

(3) $y' = \cos x$: hence the critical points are $(\frac{1}{2}\pi, 1)$, a maximum, and $(\frac{3}{2}\pi, -1)$, a minimum.

(4) $y'' = -\sin x$: the points of inflection are $(0, 0)$, with slope 1, $(\pi, 0)$, with slope -1, and $(2\pi, 0)$, with slope 1.

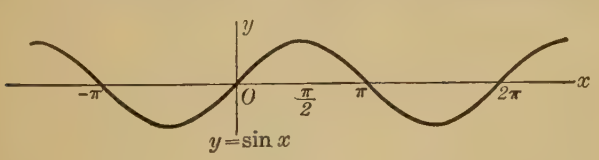


FIG. 17

The curve consists of an infinite succession of waves along the x-axis, as shown in Fig. 17.

The graphs of the cosine and tangent are also shown below;

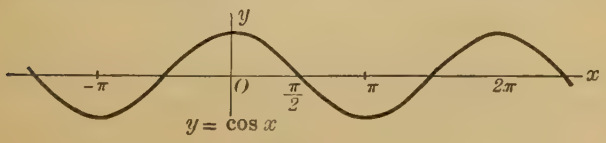


FIG. 18

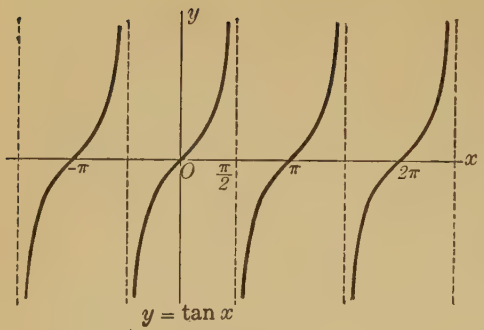


FIG. 19

they are obtained in a similar way, except that in the case of the tangent the points of discontinuity

$$x = \pm(2n + 1)\frac{\pi}{2}$$

must be specially investigated in addition to the usual discussion.

38. Curve tracing by composition of ordinates. The curve

$$y = f(x)$$

can be traced very readily if $f(x)$ has the form

$$f(x) = \phi_1(x) + \phi_2(x),$$

where

$$y = \phi_1(x), \quad y = \phi_2(x)$$

are curves whose form is easily obtained. We have only to add the ordinates of the two latter curves to obtain the required curve.

Less frequently curves may be conveniently traced by multiplying or dividing ordinates in a similar way.

Figure 20 shows the curve

$$y = 2 \sin x - \sin 2x$$

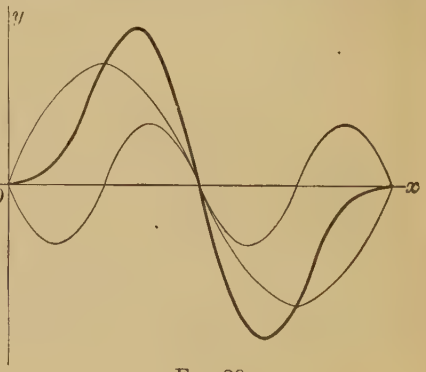


FIG. 20

traced in the interval $x = 0$ to $x = 2\pi$ by adding the ordinates of the curves

$$y = 2 \sin x, \quad y = -\sin 2x.$$

39. Graphic solution of equations. The roots of the equation

$$f(x) = 0$$

are the abscissas of the points where the curve $y = f(x)$ crosses the x-axis. Hence if we trace the curve $y = f(x)$ and measure its intercepts on Ox, we have a graphic solution of the equation $f(x) = 0$. It is usually best to get the general form of the curve by the methods of the preceding articles, and then plot it carefully, on a large scale, in the neighborhood of each of its x-intersections.

The roots of the equation

$$f(x) = \phi(x) \tag{1}$$

are the abscissas of the points of intersection of the curves

$$y = f(x), \quad y = \phi(x).$$

In case these two curves are easily traced, we thus obtain with little labor a graphic solution of (1). This method is frequently preferable to the one mentioned in the preceding paragraph.

Such methods may be useful in various ways. If no high degree of approximation is required, the graphical result may be sufficient in itself; it may be used as a rough check on a more accurate result obtained in some other way; it gives a first approximation that may be needed as a starting point for more elaborate methods, or it may suggest some value of the variable which by substitution is found to satisfy the equation exactly.

EXERCISES

Trace the following curves.

1. $y = \cos x$.
2. $y = \tan x$.
3. $y = \sec x$.
4. $y = \frac{1}{3} \sin 3x$.
5. $y = 2 \cos 2x$.
6. $y = -\cos \frac{1}{3} x$.
7. $y = -2 \sin \frac{1}{4} x$.
8. $y = \frac{1}{2} \tan 2x$.
9. $y = \sin^2 x$.
10. $y = \sin^2 2x$.
11. $y = \sin^3 x$.
12. $y = \cos^3 x$.
13. $y = 1 + \frac{1}{2} \sin 2x$.
14. $y = 1 - \cos 3x$.

15. Trace the curve $y = x + \frac{1}{x}$ by addition of ordinates.

16. Trace the curve $y = x^2 - \frac{1}{x}$ by addition of ordinates.

Trace the following curves in the interval $x = 0$ to $x = \pi$.

17. $y = \sin x + \cos x$.

18. $y = \sin 2x - \frac{1}{2}\cos 2x$.

19. $y = \frac{1}{4}\sin 2x + \frac{1}{2}\sin 3x$.

20. $y = \cos x - \frac{1}{3}\cos 3x$.

21. $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

22. $y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$.

23. $y = \sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x$.

24. $y = \sin x + \sin 2x + \frac{1}{3}\sin 3x$.

25. $y = \sin x + \sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

Trace the following curves.

26. $y = x + \sin x$.

27. $y = \cos x - 2x$.

28. $y = \cot x$.

29. $y = \csc x$.

30. $y = x \cos x$.

31. $y = x \sin x$.

32. $y = \dfrac{\sin x}{x}$.

33. $y = \dfrac{\cos x}{x}$.

Solve the following equations graphically.

34. $x^4 - 3x^3 + 3 = 0$.

35. $3x^4 - 2x^3 - 21x^2 - 4x + 11 = 0$.

36. $\cos x = x$.

37. $x + 2\cos x = 0$.

38. $x = 3\sin x$.

39. $\tan x = 2x$.

40. Maxima and minima. Many problems of the types considered in §§ 30–31 may be solved very simply by expressing the function whose maximum or minimum is to be found in terms of trigonometric functions of an angle. The method is illustrated by the following

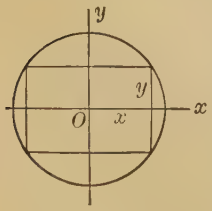


FIG. 21

Example: Find the shape of the largest rectangle that can be inscribed in a given circle (cf. the example of § 31).

The area of the rectangle is

$$A = 4xy.$$

But

$$x = a\cos\theta, \quad y = a\sin\theta,$$

so that

$$A = 4a^2\cos\theta\sin\theta.$$

Hence

$$\frac{dA}{d\theta} = 4\,a^2\,(\cos^2\theta - \sin^2\theta) = 0,$$

$$\cos\theta = \sin\theta,$$

$$y = x.$$

EXERCISES

Solve the following by use of trigonometric functions.

1. Find the shape of the rectangle of maximum perimeter inscribed in a circle. *Ans.* A square.

2. If the length of the equal sides of an isosceles triangle is given, what shape makes the area a maximum? *Ans.* Base $= 2 \times$ altitude.

3. Find the largest right circular cone that can be constructed having a given slant height. *Ans.* $r = \sqrt{2}\,h$.

4. Find the right circular cylinder of maximum volume inscribed in a given sphere. *Ans.* $h = \sqrt{2}\,r$.

5. Solve Ex. 4 if the convex surface of the cylinder is to be a maximum. *Ans.* $h = 2\,r$.

6. Assuming that the strength of a rectangular beam is proportional to the breadth and the square of the depth, find the shape of the strongest beam that can be cut from a log of given diameter.

Ans. Depth $= \sqrt{2} \times$ breadth.

7. Assuming that the stiffness of a rectangular beam is proportional to the breadth and the cube of the depth, find the shape of the stiffest beam that can be cut from a log of given diameter.

Ans. Depth $= \sqrt{3} \times$ breadth.

8. Solve Ex. 27, p. 46.

9. Find the volume of the right circular cone of maximum volume inscribed in a given sphere. *Ans.* $V = \frac{32}{81}\pi a^3$.

10. Find the right circular cone of maximum convex surface inscribed in a given sphere. *Ans.* $h = \frac{4}{3}\,a$.

11. A wall 8 ft. high is 27 ft. from a house. Find the length of the shortest ladder that will reach the house when one end rests on the ground outside the wall. *Ans.* 46.9 ft.

12. A steel girder 30 ft. long is carried along a passage 10 ft. wide and into a corridor at right angles to the passage. The thickness of the girder being neglected, how wide must the corridor be in order that the girder may go round the corner? *Ans.* 11 ft. 3 in.

13. A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of tin of width 8 in. Find the radius of the cross-section when the carrying capacity of the gutter is a maximum. *Ans.* 2.55 in.

Find the critical values between $-\pi$ and π of the following functions.

14. $y = \sin x \cos x + \sin x$. *Ans.* $x = \pm \frac{1}{3}\pi, \pm \pi$.

15. $y = \sin x \cos x - \sin x$. *Ans.* $x = 0, \pm \frac{2}{3}\pi$.

16. $y = \sin x \cos x + \cos^2 x$. *Ans.* $x = \frac{1}{8}\pi, \frac{5}{8}\pi, -\frac{3}{8}\pi, -\frac{7}{8}\pi$.

17. $y = \sin x \cos^3 x$. *Ans.* $x = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi, \pm \frac{5}{6}\pi$.

18. $y = \sin x \cos^4 x$. *Ans.* $x = \pm 90°, \pm 26° 34', \pm 153° 26'$.

19. $y = \sin^2 x + 3 \cos x$. *Ans.* $x = 0, \pm \pi$.

20. $y = 2 \sin^2 x + 3 \cos x$. *Ans.* $x = 0°, \pm 180°, \pm 41° 25'$.

21. $y = \cos \theta + 2 \sin \theta$. *Ans.* $\theta = 63° 26', -116° 34'$.

22. $y = 3 \cos 2\theta + \sin 2\theta + 6\theta$.
Ans. $\theta = 45°, -135°, 63° 26', -116° 34'$.

23. $y = \cos \theta - \cos 2\theta$. *Ans.* $\theta = 0°, \pm 180°, \pm 75° 31'$.

24. $y = \sin \theta \cos \theta - 3\theta$. *Ans.* None.

25. $y = 2 \sin \theta - \cos \theta - 2\theta$. *Ans.* $\theta = 0°, 53° 8'$.

26. $y = 2 \sin \theta + \cos \theta + 2\theta$. *Ans.* $\theta = \pm 180°, 126° 52'$.

27. $y = 5 \sin \theta + 5 \cos \theta + \theta + 1$. *Ans.* $\theta = 53° 8', -143° 8'$.

II. Inverse Trigonometric Functions

41. Inverse functions. The equation

(1) $$y = f(x)$$

defines x implicitly as a function of y; when solved for x, it takes the form

(2) $$x = \phi(y).$$

The function $\phi(y)$ is called the *inverse* of the *direct* function $f(x)$.

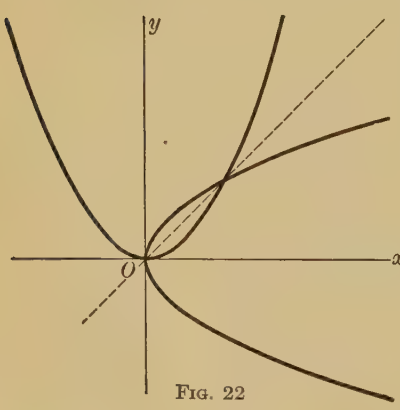


Fig. 22

For example,

(*a*) if $y = \frac{3}{2}x$, then $x = \frac{2}{3}y$;

(*b*) if $y = x^2$, then $x = \pm\sqrt{y}$;

(*c*) if $y = \pm\sqrt{a^2 - x^2}$, then $x = \pm\sqrt{a^2 - y^2}$.

In each case, the second function is the inverse of the first.

With x and y interchanged, equation (2) becomes

$$y = \phi(x).$$

Geometrically, the interchange of x and y amounts to a reflection in the line $y = x$. Hence the graph of the inverse function, with x taken as the independent variable, may be obtained from that of the direct function by reflection in the line $y = x$. This is shown in Fig. 22 for the functions $y = x^2$, $y = \pm\sqrt{x}$ (example (*b*) above).

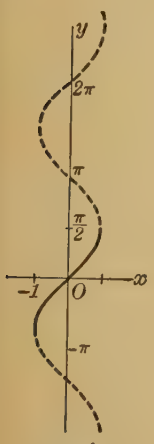
42. Inverse trigonometric functions. The symbol arcsin x, or $\sin^{-1} x$, denotes the *angle whose sine is* x:

$$y = \text{arcsin } x \text{ if } x = \sin y.$$

That is, the function arcsin x is the *inverse* of the function sin x. The graph of

$$y = \text{arcsin } x$$

is as shown in Fig. 23. It is of course the reflection of the sine curve (Fig. 17) in the line $y = x$.

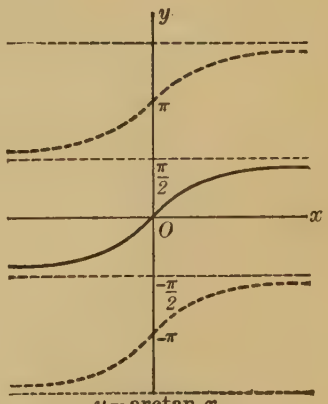


$y = \text{arcsin } x$

FIG. 23

$y = \text{arccos } x$

FIG. 24



$y = \text{arctan } x$

FIG. 25

The functions $y = \text{arccos } x$, $y = \text{arctan } x$, etc., are defined in a similar way.

43. Restriction to a single branch. When an angle is given, its sine, cosine, etc., are uniquely determined: that is, the trigonometric functions are *one-valued.* On the other hand, if the sine is given, the angle is not uniquely determined; for instance, there are infinitely many angles whose sine is $\frac{1}{2}$. Thus it appears that the inverse trigonometric functions are *infinitely many-valued:* corresponding to a given value of the variable there are infinitely many values of the function. Geometrically this means that a line $x = k$, if it meets the curve at all, meets it in an infinite number of points; the truth of this statement is evident from a glance at Figs. 23–25.

Following the rule of § 4, we shall in future, unless the contrary is noted, *confine our attention to a single branch* of each of these functions; the branch chosen is the one drawn full in each figure. Thus in our future work the function $\arcsin x$ is *restricted to the interval*

$$(1) \qquad -\frac{\pi}{2} \leqq \arcsin x \leqq \frac{\pi}{2}.$$

This means that

$$\arcsin(-1) = -\frac{\pi}{2},$$

not $\frac{3\pi}{2}$; etc. Similarly,

$$(2) \qquad -\frac{\pi}{2} \leqq \arctan x \leqq \frac{\pi}{2};$$

therefore

$$\arctan(-1) = -\frac{\pi}{4}, \quad \arctan(-\infty) = -\frac{\pi}{2}; \text{ etc.}$$

The student is urged to note these conventions carefully, since failure to observe them leads to frequent errors.

44. Inverse trigonometric equations. Equations involving only inverse trigonometric functions may in simple cases be readily transformed to algebraic form by taking the sine (or other trigonometric function) of both members of the equation. The process is best illustrated by an

Example: Solve the equation

$$\arcsin 2x - \arcsin x = \frac{\pi}{3}. \tag{1}$$

Put

$$\arcsin 2x = \alpha, \quad \arcsin x = \beta,$$

so that (1) becomes

$$\alpha - \beta = \frac{\pi}{3}. \tag{2}$$

Then

$$\sin\alpha = 2x, \quad \cos\alpha = \sqrt{1 - 4x^2},$$
$$\sin\beta = x, \quad \cos\beta = \sqrt{1 - x^2}.$$

Taking the cosine* of both members of (2), we get

$$\sqrt{1 - 4x^2}\sqrt{1 - x^2} + 2x^2 = \tfrac{1}{2},$$

or after isolating the radical, squaring, and collecting terms,

$$3x^2 = \tfrac{3}{4},$$

whence

$$x = \pm\tfrac{1}{2}.$$

Substitution in (1) shows that only the value

$$x = \tfrac{1}{2}$$

satisfies the original equation (subject to the convention (1) of § 43).

EXERCISES

Find the inverse of each of the following functions.

1. $y = 4x - 1$. *Ans.* $x = \frac{1}{4}(y + 1)$.

2. $y = -\frac{1}{2}x - \frac{1}{3}$. *Ans.* $x = -2y - \frac{2}{3}$.

3. $y = x^2 - 1$.

4. $y = x^3 - 3$.

5. $y = x^2 + ax + b$.

6. $y = (x^2 - 1)^2$.

7. $y = \dfrac{x - 1}{x + 1}$.

8. $y = \dfrac{x^2 - 1}{x^2 + 1}$.

9. Find (*a*) $\arcsin\frac{1}{2}$, (*b*) $\arcsin(-\frac{1}{2})$, (*c*) $\arcsin(-\frac{1}{2}\sqrt{2})$, (*d*) $\arctan(-\sqrt{3})$, (*e*) $\arctan\infty$, (*f*) $\arccos(-\frac{1}{2})$, (*g*) $\arccos(-1)$.

Establish the following properties of the inverse trigonometric functions.

10. $\arcsin(-x) = -\arcsin x$.

11. $\arccos(-x) = \pi - \arccos x$.

* The equation obtained by taking the sine is more complicated.

12. $\arctan(-x) = -\arctan x$.

13. $\arccos x = \frac{1}{2}\pi - \arcsin x$.

14. $\arcsin x = \arctan \dfrac{x}{\sqrt{1-x^2}}$.

15. $\arctan x = \arcsin \dfrac{x}{\sqrt{1+x^2}}$.

16. $\cos(\arctan x) = \dfrac{1}{\sqrt{1+x^2}}$.

17. $\cos(\arcsin x) = \sqrt{1-x^2}$.

18. $\sin(\arccos x) = \sqrt{1-x^2}$.

19. $\sin(2\arcsin x) = 2x\sqrt{1-x^2}$.

20. $\sin(2\arccos x) = 2x\sqrt{1-x^2}$.

21. $\cos(2\arcsin x) = 1 - 2x^2$.

22. $\cos(2\arccos x) = 2x^2 - 1$.

23. $\tan(2\arctan x) = \dfrac{2x}{1-x^2}$.

Verify the following identities.

24. $\arcsin \frac{3}{5} + \arcsin \frac{4}{5} = \frac{1}{2}\pi$.

25. $\arctan \frac{3}{4} = \arccos \frac{4}{5}$.

26. $\arctan \frac{5}{12} + \arcsin \frac{12}{13} = \frac{1}{2}\pi$.

27. $\arctan \frac{1}{3} + \arctan \frac{1}{2} = \frac{1}{4}\pi$.

28. $\arctan 2 + \arccos \frac{2}{5}\sqrt{5} = \frac{1}{2}\pi$.

29. $\arcsin \frac{9}{41} + \arctan \frac{40}{9} = \frac{1}{2}\pi$.

30. Find $\arctan \frac{1}{3} - \arctan \frac{1}{4}$. *Ans.* $\arctan \frac{1}{13}$.

31. Find $\arccos \frac{1}{3} + \frac{1}{3}\pi$. *Ans.* $\arccos(\frac{1}{6} - \frac{1}{3}\sqrt{6})$.

32. Find $\arctan \frac{1}{2} - \arctan \frac{1}{3}$. *Ans.* $\arctan \frac{1}{7}$.

33. Find $\arcsin \frac{3}{5} - \arctan \frac{3}{5}$. *Ans.* $\arctan \frac{3}{29}$.

34. Find $2\arctan \frac{1}{2} - \arctan \frac{1}{7}$. *Ans.* $\frac{1}{4}\pi$.

35. Find $2\arcsin \frac{3}{5} - \frac{1}{4}\pi$. *Ans.* $\arctan \frac{17}{31}$.

Solve the following equations.

36. $\arcsin x + \arcsin 2x = \frac{1}{2}\pi$. *Ans.* $x = \frac{1}{5}\sqrt{5}$.

37. $\arcsin 3x - \arcsin x = \frac{1}{3}\pi$. *Ans.* $x = \sqrt{\frac{3}{28}}$.

38. $\arctan 3x + \arctan 2x = \frac{1}{4}\pi$. *Ans.* $x = \frac{1}{6}$.

39. $\arcsin x - \arccos x = \frac{1}{6}\pi$. *Ans.* $x = \frac{1}{2}\sqrt{3}$.

40. $\arcsin 2x - \arccos x = \frac{1}{6}\pi$. *Ans.* $x = \frac{1}{2}$.

41. $\arcsin x = 2\arctan x$. *Ans.* $x = \pm 1, 0$.

42. $2\arcsin x + \arccos x = \frac{1}{3}\pi$. *Ans.* $x = -\frac{1}{2}$.

Transform the following equations to algebraic form.*

43. $\arcsin x + \arcsin y = \frac{1}{2}\pi$. *Ans.* $x^2 + y^2 = 1$.

44. $\arcsin x + \arcsin y = \frac{1}{3}\pi$. *Ans.* $4x^2 + 4xy + 4y^2 = 3$.

45. $\arcsin x + 2\arcsin y = \pi$. *Ans.* $4y^4 + x^2 - 4y^2 = 0$.

46. $\arctan x + \arctan y = \frac{1}{4}\pi$. *Ans.* $xy + x + y = 1$.

* If the answers to Exs. 43–48 are to hold for all values of x and y, the conventions of § 43 must be disregarded.

47. $\arctan x - \arctan y = \frac{1}{4}\pi$. *Ans.* $xy - x + y + 1 = 0$.

48. $\arctan x + 2 \arctan y = \pi$. *Ans.* $xy^2 = x + 2y$.

45. Differentiation of the inverse trigonometric functions. To differentiate the function

$$y = \arcsin x,$$

let us pass to the direct form

$$\sin y = x.$$

Differentiating by the rule for finding the derivative of an implicit function (§ 21), we find

$$\cos y \frac{dy}{dx} = 1,$$

hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos (\arcsin x)},$$

so that

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

In a similar way we find

$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}},$$

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}.$$

By formula (**5**), § 17, we find that, if u is any function of x,

(10)
$$\frac{d}{dx} \arcsin u = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}},$$

$$\frac{d}{dx} \arccos u = -\frac{\frac{du}{dx}}{\sqrt{1 - u^2}},$$

(11)
$$\frac{d}{dx} \arctan u = \frac{\frac{du}{dx}}{1 + u^2}.$$

While in the above discussion we confine our attention to a single branch of the function, it appears from Figs. 23–25 that if we know the slope at every point of one branch, we can at once find the slope at every point of any other branch.

EXERCISES

Perform the indicated differentiations.

1. $\frac{d}{dx}\arcsin 2x$.

2. $\frac{d}{dx}\arctan x^2$.

3. $\frac{d}{dy}\arcsin(1-y)$.

4. $\frac{d}{dt}\arcsin \frac{1}{3}t$.

5. $\frac{d}{dx}\arctan(1-2x)$.

6. $\frac{d}{dx}\arccos 3x$.

7. $\frac{d}{dx}\arctan \frac{\sqrt{5}\,x}{\sqrt{3}}$.

8. $\frac{d}{dt}\arctan \frac{\sqrt{2}\,t}{\sqrt{5}}$.

9. $\frac{d}{dx}\arcsin \sqrt{x}$.

10. $\frac{d}{dx}\arcsin \sqrt{1-x}$.

11. $\frac{d}{dx}\arcsin \frac{x}{\sqrt{1+x^2}}$.

12. $\frac{d}{dx}\arccos \frac{1}{\sqrt{1+x^2}}$.

Differentiate the following functions.

13. $y = x \arcsin x$.

14. $y = x \arctan x$.

15. $y = a^2 \arcsin \frac{x}{a} - x\sqrt{a^2-x^2}$.

Ans. $\frac{2x^2}{\sqrt{a^2-x^2}}$.

16. $y = \arcsin \frac{x}{a} + \frac{\sqrt{a^2-x^2}}{x}$.

Ans. $-\frac{\sqrt{a^2-x^2}}{x^2}$.

17. $y = (\arctan x)^2$.

18. $y = \sqrt{1 - \arctan 2x}$.

19. $y = x \arctan 3x$.

20. $x = t^2 \arcsin \frac{1}{2}t$.

21. $y = \frac{x}{\sqrt{a^2-x^2}} - \arcsin \frac{x}{a}$.

Ans. $\frac{x^2}{(a^2-x^2)^{\frac{3}{2}}}$.

22. $y = \arcsin \sqrt{\frac{x}{2a}}$.

Ans. $\frac{1}{2\sqrt{2ax-x^2}}$.

23. $y = \sqrt{x^2-a^2} + a \arcsin \frac{a}{x}$.

Ans. $\frac{\sqrt{x^2-a^2}}{x}$.

24. $z = (1+u^2)\arctan u$.

25. $r = \arctan\left(\frac{x}{a}\right)^3$.

26. Find the tangent to the curve $y = \arcsin x$ at the point $x = -\frac{1}{2}\sqrt{3}$.

Ans. $y + \frac{1}{3}\pi = 2(x + \frac{1}{2}\sqrt{3})$

27. Find the tangent to the curve $y = \arctan x$ at the point $x = 1$.
Ans. $y - \frac{1}{4}\pi = \frac{1}{2}(x - 1)$.

28. Find tangents to the curve $y = \arctan x$ perpendicular to the line $4x + y = 0$. *Ans.* $y \mp \frac{1}{3}\pi = \frac{1}{4}(x \mp \sqrt{3})$.

29. Find tangents to the curve $y = \arctan x$ parallel to the line $4x - 8y = 3$. *Ans.* $y \mp \frac{1}{4}\pi = \frac{1}{2}(x \mp 1)$.

30. Find the tangent to the curve $y = \arcsin 2x$ parallel to the line $2x - y = 1$. *Ans.* $y = 2x$.

CHAPTER VI

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

46. The exponential function. The number $a^n (a > 0)$ is defined in algebra for all *rational* values of n. In the calculus it becomes necessary to attach a meaning to the function

$$y = a^x \qquad (a > 0)$$

as x varies *continuously*.

Let x_0 be any given irrational number. It can be shown that when x approaches x_0 passing through rational values, the function a^x approaches a definite limit. This limit is denoted by a^{x_0}:

$$\lim_{x \to x_0} a^x = a^{x_0}.$$

The function

$$y = a^x,$$

called the *exponential function*, thus becomes defined for all values of x. It is one-valued and continuous, and obeys the laws of exponents, viz.:

(1) $$a^x \cdot a^t = a^{x+t},$$

(2) $$(a^x)^t = a^{xt}.$$

The exponential function is evidently positive for all values of x.

It will appear in § 50 that the function

$$y = e^x,$$

where $e = 2.71828 \cdots$, is of great importance in the calculus and its applications. The graph of this function is shown in Fig. 26, p. 69.

47. The logarithm. The inverse of the exponential function is the *logarithm*, defined by the statement that

$$y = \log_a x \text{ if } x = a^y \qquad (a > 1^*).$$

This function is *one-valued and continuous* for all *positive* values of x. The number a is called the *base* of the system of logarithms.

The following facts concerning the function

$$y = \log_a x$$

appear at once from the definition:

(*a*) *Negative numbers have no (real) logarithms.*

(*b*) *Numbers between* 0 *and* 1 *have negative logarithms.*

(*c*) *Numbers greater than* 1 *have positive logarithms.*

(*d*) *As* x *approaches* 0, y *becomes indefinitely large and negative.*

(*e*) *The logarithm of* 1 *is* 0.

(*f*) *As* x *increases indefinitely,* y *does likewise.*

The curve

$$y = \log_e x$$

is shown in Fig. 27. The properties listed above are most easily remembered by visualizing the graph.

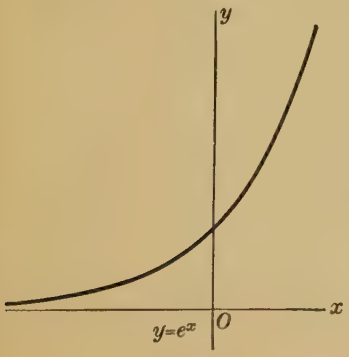


FIG. 26

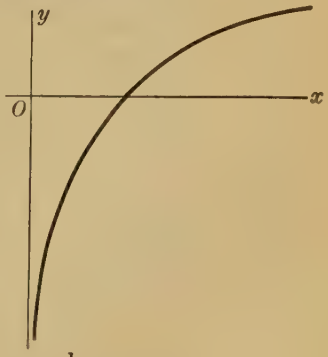


FIG. 27

* The assumption $a > 1$ is introduced for simplicity; this condition is satisfied in all cases of practical importance.

48. Fundamental properties of logarithms. Further important properties of the logarithmic function are as follows:

$$\log_a xy = \log_a x + \log_a y, \tag{1}$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y, \tag{2}$$

$$\log_a x^n = n \log_a x, \tag{3}$$

$$\log_a a^x = x, \tag{4}$$

$$a^{\log_a x} = x. \tag{5}$$

To prove (1), let

$$\log_a xy = p, \quad \log_a x = m, \quad \log_a y = n:$$

then we must show that

$$p = m + n.$$

Passing to the direct form, we have

$$xy = a^p, \quad x = a^m, \quad y = a^n,$$

or

$$a^p = a^m \cdot a^n.$$

Hence, by (1) of § 46,

$$p = m + n.$$

Formulas (2) and (3) may be proved in a similar way.

Formula (4) is merely a restatement of the definition of the logarithm; formula (5) is the converse. To prove (5), set

$$a^{\log_a x} = t,$$

and take logarithms to the base a on each side:

$$\log_a x = \log_a t,$$

whence

$$t = x.$$

49. Change of base. Given a table of logarithms to any base b, the logarithm of any number x to the base a can be found by the *formula for change of base:*

$$\log_a x = \frac{\log_b x}{\log_b a}. \tag{1}$$

To prove this formula, let

$$m = \log_a x \text{ and } n = \log_b x;$$

then

$$x = a^m = b^n$$

Taking logarithms to the base b we get

$$m \log_b a = n,$$

which gives the formula at once.

Taking $x = b$ in (1), we obtain the formula

$$\log_a b = \frac{1}{\log_b a}. \tag{2}$$

50. Natural logarithms. Only two systems of logarithms are of actual importance in practice. Logarithms to the base 10, called *common logarithms*, possess the great advantage that the "mantissa," or fractional part of the logarithm, is independent of the position of the decimal point in the given number; common logarithms are therefore used exclusively in computing. It will appear presently, however, that in the applications of the calculus it is more convenient to take as the base a certain number, denoted by e, defined by the formula

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z, \tag{1}$$

and having the value

$$e = 2.71828 \cdots.$$

(It is of course not obvious that the limit occurring in (1) exists*, but that fact is proved in more advanced texts.)

Logarithms to the base e are called *natural logarithms*. In all our future work, *when no base is indicated, the base e will be understood.*

A table of common logarithms gives

$$\log_{10} e = \log_{10} 2.71828 = 0.43429.$$

* The existence of e can be made plausible as follows. Setting

$$y = \left(1 + \frac{1}{z}\right)^z,$$

we obtain the following table of values:

z	1	2	3	10	100	1000	10000	100000
y	2	2.25	2.37037	2.59374	2.70481	2.71692	2.71815	2.71827

It is therefore reasonable to suppose that y approaches a limit but little greater than the last value given.

We thus derive from (1), § 49, the following specific formulas:

$$\begin{cases} \log_{10} x = 0.43429 \log_e x, \\ \log_e x = 2.30259 \log_{10} x. \end{cases}$$

For reference we give the natural logarithms, to five decimal places, of the prime numbers below 100, including for convenience log 10 and log 100. With this table we can obtain the logarithm of any composite number containing no prime factor greater than 100 *; logarithms of decimals can also be found as in the following

Example: $\log 0.092 = \log \dfrac{2^2.23}{10^3}$

$= 2 \log 2 + \log 23 - 3 \log 10$

$= -2.3860.$

NATURAL LOGARITHMS

N	Log	N	Log	N	Log
2	0.69315	23	3.13549	61	4.11087
3	1.09861	29	3.36730	67	4.20469
5	1.60944	31	3.43399	71	4.26268
7	1.94591	37	3.61092	73	4.29046
10	2.30259	41	3.71357	79	4.36945
11	2.39790	43	3.76120	83	4.41884
13	2.56495	47	3.85015	89	4.48864
17	2.83321	53	3.97029	97	4.57471
19	2.94444	59	4.07754	100	4.60517

EXERCISES

Find x in the following cases.

1. (*a*) $\log_{10} x = 2$; (*b*) $\log_{10} x = -\frac{1}{2}$; (*c*) $\log_{10} x = \frac{1}{3}$; (*d*) $\log_{10} x = -\frac{5}{2}$; (*e*) $\log_{10} x = 6$; (*f*) $\log_{10} x = -6$.

2. (*a*) $\log x = 3$; (*b*) $\log x = -\frac{2}{3}$; (*c*) $\log_2 x = 4$; (*d*) $\log_a x = 0$; (*e*) $\log_a x = 1$.

3. $\log x = \log 2 + \log 3$.

4. $\log x = 2 \log 2 - \log 5$. *Ans.* $\frac{4}{5}$.

5. $\log x = \frac{1}{2} \log 3 - \frac{3}{2} \log 2$. *Ans.* $\frac{1}{4}\sqrt{6}$.

* The fifth place of a logarithm thus found is slightly uncertain.

6. $\log x = \log 10 - 2 \log 5 + \log 3$. *Ans.* $\frac{6}{5}$.

7. $\log_{10} x = 2 + \log_{10} 2 - 2 \log_{10} 5$. *Ans.* 8.

8. $\log_{10} x = 6 \log_{10} 2 + \log_{10} 3 - 3$. *Ans.* 0.192.

9. $\log x = \frac{1}{2} \log 2 + \frac{1}{2} \log 3 + \frac{3}{2} \log 6 + 1$. *Ans.* $36e$.

10. $\log_{10} x = \frac{7}{2} - \frac{1}{2} \log_{10} 2 - \frac{3}{2} \log_{10} 5$. *Ans.* 200.

11. Simplify (*a*) $a^{\log 2}$, (*b*) $a^{-\log 2}$, (*c*) $a^{2 \log 3}$, (*d*) $a^{-3 \log 3}$, (*e*) $a^{1+\log 3}$, (*f*) $a^{2 \log 2 - 1}$, (*g*) $a^{2-2 \log 5}$, (*h*) $a^{x+\log x}$, the logarithms being taken to the base a in each case. *Ans.* (*b*) $\frac{1}{2}$; (*e*) $3a$.

12. Show that negative numbers have no (real) logarithms.

13. Show that numbers between 0 and 1 have negative logarithms; numbers greater than 1, positive logarithms.

14. Show that

$$\lim_{x \to 0^+} \log_a x = -\infty.$$

15. Prove formulas (2) and (3) of § 48.

Find the inverse of each of the following functions.

16. $y = 10^{2x}$. *Ans.* $x = \frac{1}{2} \log_{10} y$.

17. $y = e^{x-3}$. *Ans.* $x = 3 + \log y$.

18. $y = e^{\frac{1}{2}x^2}$. *Ans.* $x = \pm\sqrt{2 \log y}$.

19. $y = 10^{x^2-1}$. *Ans.* $x = \pm\sqrt{1 + \log_{10} y}$.

20. $y = \sin e^{2x}$. *Ans.* $x = \frac{1}{2} \log \arcsin y$.

21. $y = \cos 2e^{-x}$. *Ans.* $x = \log 2 - \log \arccos y$.

22. $y = \arctan e^x$. *Ans.* $x = \log \tan y$.

23. $y = \arcsin (e^x - 1)$. *Ans.* $x = \log (1 + \sin y)$.

24. $y = \log 3x$. *Ans.* $x = \frac{1}{3} e^y$.

25. $y = \log (x^2 - 4)$. *Ans.* $x = \pm\sqrt{e^y + 4}$.

26. $y = \log \cos x$. *Ans.* $x = \arccos e^y$.

27. $y = \log \tan^2 x$. *Ans.* $x = \pm \arctan e^{\frac{y}{2}}$.

28. $y = \log \dfrac{x-1}{x}$. *Ans.* $x = \dfrac{1}{1 - e^y}$.

Break up the following into sums and differences of simpler logarithms.

29. $\log \dfrac{x(x+1)}{x-2}$.

30. $\log \dfrac{x^2-1}{x^2}$.

31. $\log \sqrt{\dfrac{1-x}{1+x}}$.

32. $\log \dfrac{x^2 - 5x + 6}{2x^3}$.

33. $\log \dfrac{xe^x}{(x^2-1)^2}$. *Ans.* $x + \log x - 2 \log (x+1) - 2 \log (x-1)$.

34. $\log \dfrac{x^2 e^{-x}}{x-1}$.

35. $\log \sqrt[3]{\dfrac{a+bx}{x}}$.

Find x in the following cases, using a table of common logarithms.

36. $2^x = 5$. *Ans.* 2.32. **37.** $3^x = 4$. *Ans.* 1.26.

38. $6^x = 2$. *Ans.* 0.39. **39.** $2^x = 328$. *Ans.* 8.36.

40. $13^x = 7956$. *Ans.* 3.50. **41.** $5^x = 665{,}580$. *Ans.* 8.33.

42. Find the following natural logarithms: (*a*) log 68; (*b*) log 186; (*c*) $\log \frac{72}{5}$; (*d*) $\log \sqrt{102}$; (*e*) $\log \frac{78}{175}$; (*f*) $\log \frac{893}{134}$.

43. Find the following natural logarithms: (*a*) log 1.3; (*b*) log 1.86; (*c*) log 0.00776; (*d*) $\log (0.14)^3$; (*e*) $\log \sqrt[3]{0.0171}$.

44. Show that $\log (\sqrt{x+1} - \sqrt{x}) = -\log (\sqrt{x+1} + \sqrt{x})$.

51. Exponential and logarithmic equations. Equations involving exponential functions only, or logarithms only, may in simple cases be solved by applying the theory of §§ 47, 48. The method is sufficiently explained by the following

Examples: (*a*) Solve for x the equation

$$e^x - e^{-x} = 2.$$

Multiplying by e^x, we get

$$e^{2x} - 1 = 2\,e^x,$$

or

$$e^{2x} - 2\,e^x - 1 = 0.$$

This is a quadratic equation in e^x, whose solution * is

$$e^x = 1 + \sqrt{2}.$$

Hence

$$x = \log (1 + \sqrt{2}).$$

(*b*) Solve the equation

$$\log (2\,x + 7) - \log (x - 1) = \log 5.$$

Combining the logarithms in the left member†, we get

$$\log \frac{2\,x + 7}{x - 1} = \log 5,$$

* The negative sign before the radical must be rejected, since e^x is never negative.

† With a little practice this step can be omitted, equation (1) being written at once.

whence

$$\frac{2x+7}{x-1} = 5, \tag{1}$$

and

$$x = 4.$$

Check: $\log 15 - \log 3 = \log 5$.

EXERCISES

Solve the following equations.

1. $e^x - e^{-x} = 1$. *Ans.* $x = \log(1+\sqrt{5}) - \log 2$.
2. $e^x + e^{-x} = 4$. *Ans.* $x = \log(2 \pm \sqrt{3})$.
3. $e^x + 3e^{-x} = 4$. *Ans.* $x = 0, \log 3$.
4. $2e^x - 1 = 3e^{-x}$. *Ans.* $x = 0.4055$.
5. $2^x = 10^{x-1}$. *Ans.* $x = \dfrac{1}{1 - \log_{10} 2}$.
6. $2^{3x} = 6^{x-1}$. *Ans.* $x = \dfrac{\log 6}{\log 6 - 3\log 2}$.
7. $10^x = 5^{3x-4}$. *Ans.* $x = \dfrac{4\log_{10} 5}{3\log_{10} 5 - 1}$.
8. $\log(3x+2) - \log(x-4) - \log 2 = \log 5$. *Ans.* $x = 6$.
9. $\log(1-x) - \log 2 + \log(2x+3) = 0$. *Ans.* $x = \frac{1}{2}, -1$.
10. $\log x + \log(x+1) - \log(x-1) = \log 6$. *Ans.* $x = 2, 3$.
11. $\log_{10}(x+1) - \log_{10} x = 1$. *Ans.* $x = \frac{1}{9}$.
12. $\log(3x+2) - 2\log x = \log 2$. *Ans.* $x = 2$.
13. $\frac{1}{2}\log(x^2-2) + \log x - \frac{1}{2}\log 3 = 0$. *Ans.* $x = \sqrt{3}$.
14. $\log_{10} x + \log_{10}(x+10) - 2 = 0$. *Ans.* $x = 5(\sqrt{5}-1)$.
15. $\log_{10}(3x+1) - \log_{10}(x+10) + 1 = 0$. *Ans.* $x = 0$.

Write the following equations in a form free of logarithms.

16. $\log x + \log y - \log(x+y) = \log 2$. *Ans.* $xy - 2x - 2y = 0$.
17. $\log(a+x) + \log(a-x) - 2\log y = 0$. *Ans.* $x^2 + y^2 = a^2$.
18. $2\log x - \log(x+y) - \log(x-y) = \log 2$. *Ans.* $x^2 = 2y^2$.
19. $3\log_{10} x - 2\log_{10} y = 1$. *Ans.* $x^3 = 10y^2$.
20. $x + \log x - \log y = 0$. *Ans.* $y = xe^x$.
21. $x + \log \sin x - \log y = \log 2$. *Ans.* $y = \frac{1}{2}e^x \sin x$.
22. $x - y - 2\log x = 0$. *Ans.* $e^{x-y} = x^2$.

Find the inverse of each of the following functions.

23. $y = 2e^x + e^{-x}$. *Ans.* $x = \log(y \pm \sqrt{y^2 - 8}) - 2\log 2$.

24. $y = e^x - 4e^{-x}$. *Ans.* $x = \log(y + \sqrt{y^2 + 16}) - \log 2$.

25. $y = e^{2x} - e^x + 1$. *Ans.* $x = \log(\frac{1}{2} \pm \frac{1}{2}\sqrt{4y - 3})$.

26. $y = \dfrac{e^x - 1}{e^x + 1}$. *Ans.* $x = \log(1 + y) - \log(1 - y)$.

27. $y = \log(1 + x) - 2\log x$. *Ans.* $x = \dfrac{1 + \sqrt{1 + 4e^y}}{2e^y}$.

28. $y = \log(1 + \sqrt{2 - x^2})$. *Ans.* $x = \pm\sqrt{1 + 2e^y - e^{2y}}$.

29. $y = \log(1 + x^2) - \log x$. *Ans.* $x = \frac{1}{2}(e^y \pm \sqrt{e^{2y} - 4})$.

52. The derivative of the logarithm. To obtain the derivative of the logarithm we proceed by the general method of § 12:

$$\begin{aligned} y &= \log_a x, \\ y + \Delta y &= \log_a (x + \Delta x), \\ \Delta y &= \log_a (x + \Delta x) - \log_a x = \log_a \frac{x + \Delta x}{x} \\ &= \log_a \left(1 + \frac{\Delta x}{x}\right), \end{aligned}$$

by property (2) of § 48. Hence,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right).$$

Let us multiply and divide by x and then make use of (3), § 48:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{1}{x} \cdot \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) \\ &= \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}. \end{aligned}$$

Hence

$$(1) \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$(2) \qquad = \frac{1}{x} \log_a \left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right].$$

Now, putting

$$\frac{x}{\Delta x} = z,$$

we see that the limit occurring in (2) is

$$\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z = e,$$

by § 50. Hence

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e. \tag{3}$$

In case the base a of the system of logarithms is the number e, the numerical factor $\log_a e$ reduces to unity, and (3) takes the much simpler form

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

This is the reason for the use of natural logarithms in the calculus.

By formula (**5**) of § 17, if u is any function of x,

$$\frac{d}{dx} \log_a u = \frac{\frac{du}{dx}}{u} \cdot \log_a e,$$

and

$$\frac{d}{dx} \log u = \frac{\frac{du}{dx}}{u}. \tag{12}$$

Example: Find the slope of the curve

$$y = \log \sqrt{1 + 3x}$$

at the point (x, y).

Let us write y in the form

$$y = \tfrac{1}{2} \log (1 + 3x).$$

Then

$$y' = \frac{1}{2} \cdot \frac{3}{1 + 3x} = \frac{3}{2 + 6x}.$$

53. The power formula. The formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

was proved in § 16 for positive integral values of n. We are now in position to show that the formula holds for all values of n.

Setting

$$y = x^n,$$

let us take the natural logarithm of both members:

$$\log y = n \log x.$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x},$$

whence

$$\frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

EXERCISES

Differentiate the following functions.

1. $y = \log 5x$.
2. $y = \log (1 - x^2)$.
3. $y = \log\sqrt{3 - 2x}$.
4. $y = \log (2x^2 + 1)^2$.
5. $y = \log\dfrac{2x - 1}{2x + 1}$. *Ans.* $y' = \dfrac{4}{4x^2 - 1}$.
6. $y = \log\dfrac{1 - x^2}{1 + x^2}$. *Ans.* $y' = \dfrac{4x}{x^4 - 1}$.
7. $y = \log\sqrt{\dfrac{1 + x}{1 - x}}$. *Ans.* $y' = \dfrac{1}{1 - x^2}$.
8. $y = \log\dfrac{x^2}{(3 - x)^2}$. *Ans.* $y' = \dfrac{6}{x(3 - x)}$.
9. $u = \log\dfrac{x^3}{(x + 1)^2}$. *Ans.* $\dfrac{du}{dx} = \dfrac{x + 3}{x(x + 1)}$.
10. $x = \log\sqrt{\dfrac{1 - t^2}{t}}$. *Ans.* $\dfrac{dx}{dt} = \dfrac{t^2 + 1}{2t(t^2 - 1)}$.
11. $z = \log\dfrac{x(x - 3)}{x - 2}$. *Ans.* $\dfrac{dz}{dx} = \dfrac{x^2 - 4x + 6}{x(x - 3)(x - 2)}$.
12. $y = \log_{10} (x^2 - 1)$. *Ans.* $y' = \dfrac{2x}{x^2 - 1} \times 0.43429$.
13. $y = \log_a \sqrt{1 - x}$. *Ans.* $y' = \dfrac{-\log_a e}{2(1 - x)}$.

14. $y = \log_{10}(1 - v)^3$.

15. $y = \log_{10} z(z - 1)$.

16. $y = \log \sin x$.

17. $y = \log \cos 2x$.

18. $y = \log \tan^2 x$.

19. $y = \log \sec x$.

20. $y = x \log x$.

21. $y = x^2 \log(1 - x)$

22. $y = \dfrac{\log x}{x}$.

23. $y = \dfrac{x}{\log x}$.

24. $y = \log^2 x$.

Ans. $y' = \dfrac{2 \log x}{x}$.

25. $y = \log^2 \sin x$.

Ans. $y' = 2 \cot x \log \sin x$.

26. $r = \log \log x$.

Ans. $r' = \dfrac{1}{x \log x}$.

27. $F(x) = x \log^2 x$.

Ans. $F'(x) = 2 \log x + \log^2 x$.

28. If $y = \log x$, find y'', y''', $y^{(4)}$.

29. Find the tangent to the curve $y = \log x$ at (*a*) $y = 0$; (*b*) $y = -\frac{1}{2}$; (*c*) $x = e^2$.

30. Find the tangent to the curve $y = \log x$ parallel to the line $3x - y = 2$.

Ans. $3x - y = 1 + \log 3$.

31. Find the equation of a tangent to the curve $y = x \log x$ parallel to the line $3x - 2y = 5$.

32. In passing from (1) to (2), § 52, we make use of the principle that

$$\lim (\log x) = \log (\lim x).$$

From what property of the logarithm does this principle follow?

54. Differentiation of the exponential function. The derivative of the exponential function a^x may be found as follows.

If

$$y = a^x,$$

then

(1) $$\log_a y = x.$$

Differentiating (1) by the rule for implicit functions (§ 21), we find

$$\frac{1}{y}\frac{dy}{dx}\log_a e = 1,$$

$$\frac{dy}{dx} = \frac{y}{\log_a e} = y \log_e a,$$

by (2), § 49; hence

$$\frac{d}{dx}a^x = a^x \log_e a.$$

For the case $a = e$, the result is

$$\frac{d}{dx} e^x = e^x.$$

If u is a function of x, these formulas become, respectively,

$$\frac{d}{dx} a^u = a^u \log_e a \cdot \frac{du}{dx},$$

(13)
$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

Example: $\dfrac{d}{dx} e^{\frac{x^3}{3}} = e^{\frac{x^3}{3}} \cdot \dfrac{3\,x^2}{3} = x^2\, e^{\frac{x^3}{3}}.$

EXERCISES

Differentiate the following functions.

1. $y = e^{3x}$. *Ans.* $y' = 3\,e^{3x}$.
2. $y = e^{-\frac{1}{2}x^2}$. *Ans.* $y' = -\,xe^{-\frac{1}{2}x^2}$.
3. $y = 3\,xe^{-x}$. *Ans.* $y' = 3\,e^{-x}(1 - x)$.
4. $y = \dfrac{e^{2x}}{x^2}$. *Ans.* $y' = \dfrac{2\,e^{2x}(x - 1)}{x^3}$.
5. $r = e^{-\theta} \sin 2\,\theta$. *Ans.* $\dfrac{dr}{d\theta} = e^{-\theta}(2 \cos 2\,\theta - \sin 2\,\theta)$.
6. $u = e^t \cos 3\,t$.
7. $y = e^{\sin x}$.
8. $y = e^x \log x$.
9. $y = \dfrac{e^{-x}}{x - 1}$.
10. $y = 2^{-3x}$. *Ans.* $y' = -\,3 \cdot 2^{-3x} \cdot \log 2$.
11. $y = 10^{2x}$. *Ans.* $y' = 2 \cdot 10^{2x} \cdot \log 10$.
12. $y = \log (e^x + 1)$. *Ans.* $y' = \dfrac{e^x}{e^x + 1}$.
13. $y = (1 - e^{2x})^4$. *Ans.* $y' = -\,8(1 - e^{2x})^3 e^{2x}$.
14. $r = \tan^2 e^{2\theta}$. *Ans.* $\dfrac{dr}{d\theta} = 4\,e^{2\theta} \tan e^{2\theta} \sec^2 e^{2\theta}$.
15. $y = e^{2x}(1 - x)^2$.
16. $z = \dfrac{e^v}{e^v - 1}$.
17. $y = \log \dfrac{e^x}{x}$. *Ans.* $y' = 1 - \dfrac{1}{x}$.
18. $z = \log \dfrac{e^v}{e^v - 1}$. *Ans.* $\dfrac{dz}{dv} = \dfrac{1}{1 - e^v}$.
19. $y = \sin^2 e^{2x}$.
20. $y = \sqrt{1 + e^{2x}}$.
21. $u = 3\,e^{-x} \tan \frac{1}{2}\,x$.
22. $y = e^z \sqrt{1 - z}$.
23. If $y = e^{ax}$, find y', y'', $\cdots$, $y^{(n)}$.
24. If $y = xe^x$, find y', y'', $\cdots$, $y^{(n)}$.

Trace the following curves.

25. $y = \frac{1}{2} e^{-2x}$. **26.** $y = 3 e^{3x}$. **27.** $y = \log (1 - x)$.

28. $y = \log (2x + 3)$. **29.** $y = \log (x - 4)$. **30.** $y = x + e^x$.

31. $y = \log x - x$. **32.** $y = \log \frac{1 + x}{1 - x}$. **33.** $y = \log (x^2 - 1)$.

34. $y = \log (x - x^2)$. **35.** $y = e^x - 2 e^{-x}$. **36.** $y = \log^2 x$.

Solve the following equations.

37. $x + e^x = 0$. **38.** $x + \log_{10} x = 0$. **39.** $2 xe^x = 1$.

55. Hyperbolic functions. A class of exponential functions of frequent occurrence in some applications are known as *hyperbolic functions.* They are denoted by the symbols $\sinh x$ (read hyperbolic sine of x), $\cosh x$, and $\tanh x$, and are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}.$$

The reciprocals of these are $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ respectively.

The inverses of the hyperbolic functions are called *anti-hyperbolic functions:*

$$y = \sinh^{-1} x \text{ if } x = \sinh y, \text{ etc.}$$

The elementary properties of the hyperbolic functions are easily obtained from the definitions; their derivation is left to the student in the exercises below.

EXERCISES

Establish the following properties of the hyperbolic functions.

1. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

2. $\sinh (-x) = -\sinh x$, $\cosh (-x) = \cosh x$, $\tanh (-x) = -\tanh x$.

3. $\cosh^2 x - \sinh^2 x = 1$.

4. $\operatorname{sech}^2 x = 1 - \tanh^2 x$.

5. $\operatorname{csch}^2 x = \coth^2 x - 1$.

6. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$

7. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$

8. $\sinh 2x = 2 \sinh x \cosh x.$

9. $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x.$

10. $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}.$

11. $\dfrac{d}{dx}\sinh x = \cosh x.$

12. $\dfrac{d}{dx}\cosh x = \sinh x.$

13. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x.$

14. Taking

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

show that

$$y = \sinh^{-1} x = \log\left(x + \sqrt{1+x^2}\right).$$

Establish the following formulas.

15. $\cosh^{-1} x = \log\left(x \pm \sqrt{x^2-1}\right), \qquad x \geqq 1.$

16. $\tanh^{-1} x = \tfrac{1}{2}\log\dfrac{1+x}{1-x}, \qquad -1 < x < 1.$

17. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}.$

18. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{\pm 1}{\sqrt{x^2-1}}.$

19. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}.$

Differentiate the following functions.

20. $y = \sinh 3x.$

21. $y = \sqrt{1 - \cosh 3x}.$

22. $y = \tanh 3x^2.$

23. $u = \sinh^2 2x.$

24. $r = (1 + \tanh 2z)^3.$

25. $y = x\cosh^2 2x.$

26. $y = e^{-x}\sinh 2x.$

27. $y = \log\tanh x.$

28. $y = \log\cosh x.$

29. $y = \log\sinh 2x.$

30. Trace the curve $y = \sinh x$ (cf. § 38).

31. The *catenary* is the curve in which a homogeneous cord or chain hangs when suspended from two of its points under its own weight. The equation is

$$y = a\cosh\frac{x}{a}.$$

Trace the curve.

32. Trace the curve $y = \tanh x$.

Fundamental Differentiation Formulas

(1) $$\frac{dc}{dx} = 0,$$

(2) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

(3) $$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx},$$

(3′) $$\frac{d}{dx}cv = c\frac{dv}{dx},$$

(4) $$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$

(4′) $$\frac{d}{dx}\frac{c}{v} = -\frac{c}{v^2}\frac{dv}{dx},$$

(5) $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$

(6) $$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$$

(6′) $$\frac{d}{dx}\sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}},$$

(7) $$\frac{d}{dx}\sin u = \cos u\frac{du}{dx},$$

(8) $$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx},$$

(9) $$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx},$$

(10) $$\frac{d}{dx}\arcsin u = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}},$$

(11) $$\frac{d}{dx}\arctan u = \frac{\frac{du}{dx}}{1 + u^2},$$

$$(12) \qquad \frac{d}{dx}\log u = \frac{\frac{du}{dx}}{u},$$

$$(13) \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

MISCELLANEOUS EXERCISES

Differentiate the following functions.

1. $y = (x+1)^3(2x-1)^2$.
2. $y = x^2(1-x^2)^3$.
3. $y = \dfrac{x^3}{(1-x)^3}$.
4. $y = \log\dfrac{x^2}{(x-1)^2}$.
5. $\phi = \sin\theta\tan\theta$.
6. $y = x^3e^{-x}$.
7. $\psi = \arcsin(2x-3)$.
8. $y = z\sqrt{1-z^2}$.
9. $r = \sin^3\frac{1}{3}\theta$.
10. $y = \log\tan 3x$.
11. $y = \arctan x^3$.
12. $y = (1-x)^2(2x+3)^3$.
13. $y = e^{\cos x}$.
14. $t = \sqrt{1-\cot x}$.
15. $z = x\arcsin\dfrac{x}{2}$.
16. $u = \dfrac{x}{\sqrt{1-x^2}}$.
17. $x = \log^2\sin\theta$.
18. $x = \arctan(1-t^2)$.
19. $y = \arcsin\dfrac{x}{4}$.
20. $u = \dfrac{(3x^2-4)^2}{x^2+1}$.
21. $y = \cos^4 2x$.
22. $u = \tan^2(1-x)$.
23. $y = \log\log\tan x$.
24. $y = \sinh^4 x$.
25. $y = \dfrac{\sqrt{3-4x}}{2}$.
26. $\phi = \cos^2\left(\dfrac{\pi}{4}-\theta\right)$.
27. $y = 2^{2x}$.
28. $y = \sqrt{\sin x^2}$.
29. $y = x\log\sqrt{1-x}$.
30. $y = (e^{2x}-1)^4$.
31. $y = \sin x\cos 2x$.
32. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.
33. $y = \dfrac{(x-1)^3}{(x^2+3x+3)^2}$.
34. $r = \dfrac{\sin^2 2\theta}{(1-\cos 2\theta)^2}$.
35. $y = \log(e^{2x}+1)$.
36. $y = \log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.
37. $y = \cosh^2 x$.
38. $y = \log\tanh 2x$.
39. $y = \dfrac{a}{2}\cosh\dfrac{x}{a}$.
40. $y = \log\sqrt{\dfrac{1-\sin x}{1+\sin x}}$.

CHAPTER VII

THE DIFFERENTIAL

56. Infinitesimals. An *infinitesimal* is *a variable whose limit is* 0. Such variables are of importance in the calculus for various reasons, but chiefly because in the problem of differentiation Δx and Δy are infinitesimals.

It should be noted that a constant, however small, is not an infinitesimal. Further, an infinitesimal is not necessarily small at all stages of its variation; the only thing necessary is that *ultimately* it must become and remain numerically less than any positive assignable constant.

If one infinitesimal is a function of another, the independent variable is called the *principal infinitesimal.* Thus in differentiation Δy is a function of the principal infinitesimal Δx.

57. Order of infinitesimals. A concept of fundamental importance in the study of infinitesimals is that of *order.* Let an infinitesimal v be defined as a function of a principal infinitesimal u. If

$$\lim_{u \to 0} \frac{v}{u} = k, \tag{1}$$

where $k \neq 0$, then u and v are said to be *infinitesimal of the same order;* the same idea is expressed by saying that v is *infinitesimal of the first order* with respect to u. For example, $\sin \theta$ is infinitesimal of the first order with respect to θ, since by § 35

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Further, if

$$\lim_{u \to 0} \frac{v}{u} = 0, \tag{2}$$

v is said to be of *higher order* than u. More precisely, if a number n can be found such that

$$\lim_{u \to 0} \frac{v}{u^n} = k, \tag{3}$$

where $k \neq 0$, v is said to be *infinitesimal of the n-th order* with respect to u. For example, the quantity $1 - \cos^2 \theta$ is evidently infinitesimal with respect to θ. Now we may write

$$1 - \cos^2 \theta = \sin^2 \theta,$$

so that

$$\lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{\theta^2} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2} = 1;$$

it follows from (3) that $1 - \cos^2 \theta$ is infinitesimal of the second order with respect to θ.

58. Principal part of an infinitesimal. If two infinitesimals u and v are of the same order, it follows from (1), § 57, that

$$v = ku + \epsilon, \tag{1}$$

where ϵ is an infinitesimal (or a group of infinitesimals) of higher order. By (2), § 57,

$$\lim \frac{\epsilon}{u} = 0.$$

Hence for sufficiently small values of u the term ku, although actually small, is relatively large as compared with ϵ. For this reason the term ku — *i.e.* the term of lowest order — is called the *principal part* of v.

Examples: (*a*) When the side l of a square increases by an amount Δl, the area increases by an amount

$$\Delta A = (l + \Delta l)^2 - l^2 = 2\,l\Delta l + \overline{\Delta l}^2.$$

(In the notation of formula (1) above, $u = \Delta l$, $v = \Delta A$, $k = 2\,l$, $\epsilon = \overline{\Delta l}^2$.) If Δl approaches 0, ΔA does also. The

two infinitesimals are of the same order, since

$$\lim_{\Delta l \to 0} \frac{\Delta A}{\Delta l} = \lim_{\Delta l \to 0} (2\,l + \Delta l) = 2\,l.$$

The principal part of ΔA is $2\,l\Delta l$. The figure illustrates the fact that ΔA consists of a term of the first and a term of the second order.

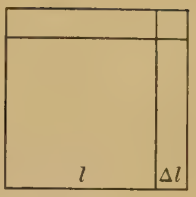


FIG. 28

(*b*) If $y = x^3 + 2\,x^2$,

then

$$\begin{aligned}\Delta y &= 3\,x^2\Delta x + 3\,x\overline{\Delta x}^2 + \overline{\Delta x}^3 + 4\,x\Delta x + 2\,\overline{\Delta x}^2 \\ &= (3\,x^2 + 4\,x)\Delta x + (3\,x + 2)\overline{\Delta x}^2 + \overline{\Delta x}^3.\end{aligned}$$

Thus Δy contains infinitesimals of the first, second, and third orders; the principal part of course is $(3\,x^2 + 4\,x)\Delta x$.

EXERCISES

Find the order of each of the following infinitesimals.

1. (*a*) The circumference, (*b*) the area, of a circle whose radius approaches 0. *Ans.* (*a*) First; (*b*) second.

2. (*a*) The area, (*b*) the length of a diagonal, of a square whose side approaches 0. *Ans.* (*a*) Second; (*b*) first.

3. (*a*) The surface, (*b*) the volume, (*c*) the length of a diagonal, of a cube whose edge approaches 0.

4. (*a*) The surface, (*b*) the volume, of a sphere whose radius approaches 0.

5. (*a*) The volume, (*b*) the total surface, of a circular cylinder whose radius approaches 0.

6. (*a*) The volume, (*b*) the convex surface, of a circular cylinder whose altitude approaches 0. Is the total surface infinitesimal?

7. (*a*) The volume, (*b*) the total surface, of a circular cone whose radius approaches 0. *Ans.* (*b*) First.

8. $\tan\theta$, when $\theta \to 0$. *Ans.* First.

9. $\sin 3\,\theta$, when $\theta \to 0$. *Ans.* First.

10. $\sin\theta \tan 2\,\theta$, when $\theta \to 0$. *Ans.* Second.

11. $1 - \sec^2\theta$, when $\theta \to 0$. *Ans.* Second.

12. $1 - \cos^4\theta$, when $\theta \to 0$. *Ans.* Second.

13. $\sin\theta + 3\tan\theta$, when $\theta \to 0$. *Ans.* First.

14. $\sin\theta + 2\sin^2\theta - 5\sin^3\theta$, when $\theta \to 0$. *Ans.* First.

15. $1 + \sin\theta - \cos\theta$, when $\theta \to 0$. *Ans.* First.

16. $3\tan^2\theta - \tan^3\theta$, when $\theta \to 0$. *Ans.* Second.

17. $\sin\theta - \tan\theta$, when $\theta \to 0$. *Ans.* Third.

18. $2\sin\theta - \tan\theta$, when $\theta \to 0$. *Ans.* First.

19. $\sin^2\theta - \tan^2\theta$, when $\theta \to 0$. *Ans.* Fourth.

20. $\sin^4\theta - \tan^4\theta$, when $\theta \to 0$. *Ans.* Sixth.

21. $\sqrt{1+x} - \sqrt{1-x}$, when $x \to 0$. *Ans.* First.

22. $\sqrt{1+x^2} - \sqrt{1-x^2}$, when $x \to 0$. *Ans.* Second.

23. $\sqrt{1+x^n} - \sqrt{1-x^n}$, when $x \to 0$. *Ans.* n-th.

24. Is the sum of two infinitesimals itself infinitesimal? Is the product? Is the quotient?

25. Prove that the sum of two positive infinitesimals of the same order is another infinitesimal of the same order, and that their difference is of the same or higher order. Give examples.

26. In example (*a*), § 58, taking $l = 1$, tabulate the values of $ku(= 2l\Delta l)$ and $\epsilon(=\overline{\Delta l}^2)$ for $\Delta l = 10,\ 1,\ 0.1,\ 0.01,\ 0.001,\ 0.000001$.

27. Given a cube of edge l, show that if l increases by Δl, the volume increases by

$$\Delta V = 3l^2\Delta l + 3l\overline{\Delta l}^2 + \overline{\Delta l}^3.$$

Taking $l = 1$, tabulate the values of the three terms in ΔV, for $\Delta l = 10,\ 1,\ 0.1,\ 0.001,\ 0.000001$.

28. Find the change in the volume of a circular cylinder when the radius changes; when the altitude changes.

29. Find the change in the volume of a sphere when the radius changes.

In the following cases, find Δy when x changes by Δx, and indicate the principal part of Δy.

30. $y = 2x^2 - x$. *Ans.* $\Delta y = (4x - 1)\Delta x + 2\overline{\Delta x}^2$.

31. $y = 1 - x - x^2$. *Ans.* $\Delta y = (-1 - 2x)\Delta x - \overline{\Delta x}^2$.

32. $y = x^3 - 3x^2$. *Ans.* $\Delta y = (3x^2 - 6x)\Delta x + (3x - 3)\overline{\Delta x}^2 + \overline{\Delta x}^3$.

33. $y = x^3 - 4x^2 + 3x$.

Ans. $\Delta y = (3x^2 - 8x + 3)\Delta x + (3x - 4)\overline{\Delta x}^2 + \overline{\Delta x}^3$.

34. $y = x^4 - 2x^2$.

Ans. $\Delta y = (4x^3 - 4x)\Delta x + (6x^2 - 2)\overline{\Delta x}^2 + 4x\overline{\Delta x}^3 + \overline{\Delta x}^4$.

35. $y = \dfrac{1}{x}$. *Ans.* $\Delta y = -\dfrac{\Delta x}{x^2} + \dfrac{\overline{\Delta x}^2}{x^2(x + \Delta x)}$.

59. The differential. Since

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y',$$

where the derivative y' is in general different from 0, it follows that Δy and Δx are in general infinitesimals of the same order. Further, the principal part of Δy is * $y'\Delta x$.

The principal part of Δy is called the differential † of y, and is written dy:

$$dy = y'\Delta x. \tag{1}$$

Thus the increment Δy consists in general of the differential dy plus infinitesimals of higher order.

The geometric interpretation of the differential is obvious. The slope at the point $P:(x, y)$ on the curve

$$y = f(x)$$

is

$$\frac{QR}{PQ} = \frac{QR}{\Delta x} = y',$$

so that

$$QR = y'\Delta x.$$

That is, the differential dy is the segment of Δy that is intercepted by the tangent at P.

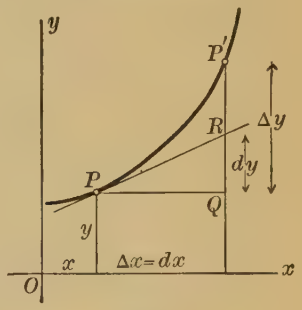


FIG. 29

To obtain a definition for the differential of the independent variable x, let us apply (1) to the function

$$y = x.$$

In this case

$$y' = 1,$$

whence

$$dy = dx = \Delta x.$$

That is, *the differential of the independent variable is equal to the increment of the variable.*

* Cf. example (*b*), § 58, and Exs. 30–35, p. 88.

† In case $y' \neq 0$. If $y' = 0$, then $dy = 0$.

We may therefore write

$$dy = y'dx,$$

and restate the definition * as follows:

The **differential** *of any function is equal to its derivative multiplied by the differential of the independent variable.*

It follows from this definition that *all the fundamental formulas for derivatives become differential formulas if we merely multiply through by* dx. For instance, the formula for the differential of a product is

$$d(uv) = udv + vdu:$$

in words, the differential of the product of two functions is equal to the first function times the differential of the second plus the second times the differential of the first.

Examples: (*a*) If

$$y = \sin 2\theta,$$

then

$$dy = 2 \cos 2\theta \, d\theta.$$

(*b*) If

$$r = \frac{z}{1 + z^2},$$

then

$$dr = \frac{(1 + z^2)dz - z \cdot 2\,zdz}{(1 + z^2)^2} = \frac{(1 - z^2)dz}{(1 + z^2)^2}.$$

(*c*) If

$$x^3 + y^3 - 3\,axy = 0,$$

then

$$3\,x^2dx + 3\,y^2dy - 3\,axdy - 3\,aydx = 0,$$
$$(y^2 - ax)dy = -(x^2 - ay)dx,$$
$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}.$$

* The derivative of y with respect to x may now be thought of as a quotient — the differential of y divided by the differential of x. This is the reason for using the symbol $\frac{dy}{dx}$ to denote the derivative. The symbol $\frac{dy}{dx}$ may thus be considered as representing an actual division — the ratio $dy \div dx$. It must be kept clearly in mind, however, that the derivative has the same meaning as before, viz.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

EXERCISES

Find the differential of each of the following functions.

1. $y = 2x^2 + 1$. *Ans.* $dy = 4x\,dx$.
2. $x = 3t^3 + t^2$. *Ans.* $dx = 9t^2dt + 2t\,dt$.
3. $y = z^3 - 2z$.
4. $x = 2t^2 - 3t + 5$.
5. $u = 2x^4 - 3x^3 + 6x$.
6. $y = t^4 - 3t^2 + 1$.
7. $y = (x - 1)^3$.
8. $y = (2x + 3)^4$.
9. $z = \sqrt{1 - y}$. *Ans.* $dz = \dfrac{-dy}{2\sqrt{1 - y}}$.
10. $w = \sqrt{a^2 - y^2}$. *Ans.* $dw = \dfrac{-y\,dy}{\sqrt{a^2 - y^2}}$.
11. $x = \cos 3t$.
12. $r = a \tan \theta$.
13. $u = x \sin x$. *Ans.* $du = x \cos x\,dx + \sin x\,dx$.
14. $y = t \cos 2t$.
15. $\phi = \theta \tan \theta$.
16. $y = \dfrac{\sin \theta}{\theta}$. *Ans.* $dy = \dfrac{\theta \cos \theta\,d\theta - \sin \theta\,d\theta}{\theta^2}$.
17. $y = \cos^2 \theta$.
18. $r = a(1 - \cos \theta)$.
19. $y = \dfrac{z}{z + 1}$. *Ans.* $dy = \dfrac{dz}{(z + 1)^2}$.
20. $r = \tan^2 2\phi$. *Ans.* $dr = 4 \tan 2\phi \sec^2 2\phi\,d\phi$.
21. $x = \dfrac{t^2 - 1}{2t}$. *Ans.* $dx = \dfrac{(t^2 + 1)dt}{2t^2}$.
22. $y = \dfrac{1}{\sqrt{1 - x}}$. *Ans.* $dy = \dfrac{dx}{2(1 - x)^{\frac{3}{2}}}$.
23. $y = \log (3x - 2)$. *Ans.* $dy = \dfrac{3\,dx}{3x - 2}$.
24. $u = \arcsin 3y$. *Ans.* $du = \dfrac{3\,dy}{\sqrt{1 - 9y^2}}$.
25. $y = e^{4x}$.
26. $y = z^2e^{2z}$.
27. $\phi = \sin (\frac{1}{4}\pi - 2\theta)$.
28. $y = x + \arctan x$. *Ans.* $du = \dfrac{(2y^2 - 1)dy}{\sqrt{y^2 - 1}}$.
29. $u = y\sqrt{y^2 - 1}$.
30. $s = \log \dfrac{z^2 + 1}{2z - 1}$. *Ans.* $ds = \dfrac{2(z^2 - z - 1)dz}{(z^2 + 1)(2z - 1)}$.
31. $u = \dfrac{a^2}{\sqrt{a^2 - x^2}}$.
32. $\theta = \arcsin \sqrt{x}$.
33. $y = \cosh 2x$.
34. $y = a \tanh \dfrac{x}{a}$.
35. $u = x \sinh x$.
36. $v = \dfrac{\cosh x}{x}$.

Find $\frac{dy}{dx}$, using differentials.

37. $x^2 - xy + y^2 = a^2$. *Ans.* $\frac{dy}{dx} = -\frac{2x - y}{2y - x}$.

38. $2x^2 - 3xy - y^2 - 2y = 0$. *Ans.* $y' = \frac{4x - 3y}{3x + 2y + 2}$.

39. $y^2 - 3x + 5y = 7$.

40. $x^3 - x^2y + y^3 = a^3$.

41. $y^3 = x^2 + 3x - 1$.

42. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

43. $x^{\frac{1}{3}} + y^{\frac{1}{3}} = a^{\frac{1}{3}}$.

44. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

45. $xy^2 + x^2y = a^3$.

46. $y = \log\sqrt{x^2 + y^2}$.

47. $\log\frac{x}{y} = x + y$. *Ans.* $y' = \frac{y(1 - x)}{x(1 + y)}$.

48. Show that $d\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}dx$, $d\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}dx$.

60. Approximate formulas. Very often we wish to compute the change in the value of a function caused by a small change in the value of the independent variable. Since Δy is equal to dy plus infinitesimals of higher order, it follows that when Δx is small, dy and Δy are, in general, nearly equal, and in many cases *the value of dy furnishes a sufficiently good approximation to the value of Δy.*

In any approximate computation, the amount by which the computed value of the function differs from the true value is called the *error* of the computation. Of course in using any approximate formula, we must make sure that the error committed is within the allowable limit of error for the problem in hand. This question will be considered more fully in Chap. XXII (see particularly § 157).

Examples: (*a*) Find an approximate formula for the area of a narrow circular ring.

The area of a circle of radius r is

$$A = \pi r^2.$$

If the radius be increased by an amount Δr, the area is increased by an amount ΔA whose principal part is

$$dA = 2\pi r dr = 2\pi r \Delta r.$$

(Since r is the independent variable, $dr = \Delta r$.) Hence the

area of a narrow circular ring is approximately the product of the circumference * by the width.

(*b*) An angle is to be read from an instrument, and its cosine computed. Find the effect upon the computed value of the cosine caused by a small error in the reading.

Since

$$d(\cos\theta) = -\sin\theta\, d\theta = -\sin\theta\,\Delta\theta,$$

the error in the value of $\cos\theta$ caused by a small error $\Delta\theta$ in the value of θ is approximately $-\sin\theta\,\Delta\theta$.

(*c*) If the possible error in the reading is 10′, and the allowable error in the cosine is 0.0005, within what range of (acute) angles can the process of example (*b*) be used?

Here we must have (numerically)

$$\sin\theta\,\Delta\theta < 0.0005,$$

$$\sin\theta < \frac{0.0005}{\Delta\theta}.$$

But

$$\Delta\theta = 10' = \frac{1}{6}\cdot\frac{\pi}{180},$$

whence

$$\sin\theta < 0.1719,$$

$$\theta < 9^\circ\ 50', \text{ approximately.}$$

EXERCISES

1. Find an approximate formula for the volume of a thin cylindrical shell of given height. *Ans.* Circumference × height × thickness.

2. Find approximately the volume of a thin spherical shell.

3. The radius of a circle is to be measured, and its area computed. If the area must be accurate to 0.1 sq. in., and the radius can be measured to 0.001 in., find the greatest radius for which the process can be used. *Ans.* Nearly 16 in.

4. The radius of a sphere is to be measured, and its volume computed. If the possible error in the radius is 0.001 in., and the allowable error in the volume is 1 cu. in., find the greatest radius for which the process can be used. *Ans.* About 9 in.

5. Show that the change in the circumference of a circle produced by a change in the radius is independent of the radius.

* Either the inner or the outer circumference.

6. Find approximately the volume of wood required to make a cubical box of edge 3 ft., using $\frac{1}{2}$-inch boards. *Ans.* $2\frac{1}{4}$ cu. ft.

7. For angles in the vicinity of 45°, find approximately the change (*a*) in $\sin\theta$, (*b*) in $\cos\theta$, (*c*) in $\tan\theta$, produced by a small change in θ. *Ans.* (*a*) $0.7\,\Delta\theta$; (*b*) $-0.7\,\Delta\theta$; (*c*) $2\,\Delta\theta$.

8. Solve Ex. 7 for angles near 30°.

9. By pacing, a man finds the side of a square field to be 1260 ft. If his measurement may be in error by as much as 2 paces in 100, what is the area of the field? *Ans.* 36.4 ± 1.5 acres.

10. Solve example (*c*) of § 60, (*a*) if the possible error in the reading is 30′ and the allowable error in the cosine is 0.005; (*b*) if the possible error in the reading is 20″ and the allowable error in the cosine is 0.0001. *Ans.* (*a*) $\theta < 35°$; (*b*) all angles.

11. An angle is to be measured and its sine computed. If the extreme errors in the angle and the sine are respectively 20′ and 0.005, within what range is the process valid? *Ans.* $\theta > 30°\ 45'$.

12. An angle is to be measured and its tangent computed. If the extreme errors in the angle and the tangent are respectively 10′ and 0.1, within what range is the work valid? Show that near 90° a small error in the angle causes a large error in the tangent. *Ans.* $\theta < 80°\ 10'$.

13. The base of a right triangle is 3 ft., the altitude 4 ft. Find approximately the change in the hypotenuse when the base changes by 1 in. *Ans.* 0.6 in.

14. The base of a pyramid is a square of side 3 ft.; the height is 5 ft. Find approximately the increase in the volume of the pyramid when the side of the base increases by $\frac{1}{4}$ in. *Ans.* 360 cu. in.

15. Find approximately the change in the reciprocal of a number produced by a small change in the number. Investigate the case when the number itself is small. *Ans.* $\dfrac{-\Delta x}{x^2}$.

16. The volume of a body of gas is measured; the pressure is then computed by the formula

$$p = \frac{k}{v}.$$

If the allowable error in p is 0.001 k, and v can be measured to 0.1 cu. ft., what is the size of the smallest container to which the process can be applied? *Ans.* 10 cu. ft.

17. In Ex. 16, the gas is contained in a cubical vessel of edge l. Find approximately the error in the value of p due to a small error in the measurement of l. *Ans.* $\dfrac{-3\,k\Delta l}{l^4}$.

18. In Ex. 17, if $l = 9.86 \pm 0.02$, how accurately can p be determined? *Ans.* 0.000006 k.

19. The attraction between two magnetic poles is inversely proportional to the square of the distance between them:

$$F = \frac{k}{r^2}.$$

If the distance is slightly increased, how is the attraction affected?

Ans. Diminished by $\frac{2\,k\Delta r}{r^3}$.

20. The attraction between two magnetic poles is measured, and the distance between them computed (cf. Ex. 19). If $k = 1$ and $F = 6.3 \pm 0.1$, find r. *Ans.* 0.398 ∓ 0.003.

21. In computing a table of square roots, the allowable error in the roots is 0.01. For what values of x may $\sqrt{x+1}$ be taken equal to $\sqrt{x}$?

Ans. $x > 2500$.

22. Find approximately the change in $\sqrt[3]{x}$ caused by a small change in x. If the allowable error in a table of cube roots is 0.005, for what values of x may $\sqrt[3]{x+1}$ be taken equal to $\sqrt[3]{x}$? *Ans.* $\frac{\Delta x}{3\,x^{\frac{2}{3}}}$; $x > 544$.

23. The tangent of an angle is measured, and the angle is then determined. Find the error in the angle produced by a small error in the tangent, (*a*) for angles near 45°; (*b*) for angles near 60°.

Ans. (*a*) $\frac{1}{2}\Delta x$; (*b*) $\frac{1}{4}\Delta x$.

24. In Ex. 23, if $\tan\theta = 0.74 \pm 0.02$, find θ. *Ans.* $36°\,30' \pm 45'$.

25. In Ex. 23, if the uncertainty in the tangent is 0.01, and the angle must be determined with an uncertainty not greater than 30′, within what range of angles may the process be employed? *Ans.* $\theta > 21°$.

26. The sine of an angle is measured, and the angle is then determined. Find the effect of a small error in the sine. In what vicinity does a small error in the sine cause a large error in the angle? *Ans.* $\frac{\Delta x}{\sqrt{1-x^2}}$.

27. In Ex. 26, if the sine is uncertain by 0.0005, and the allowable error in the angle is 10′, for what angles can the computation be made?

Ans. $\theta < 80°$.

28. Find approximately the difference between $\log(x+1)$ and $\log x$ when x is large. *Ans.* $\frac{1}{x}$.

Using the table, p. 72, find the following logarithms.

29. log 101.	*Ans.* 4.615.	**30.** log 127.	*Ans.* 4.844.
31. log 359.	*Ans.* 5.883.	**32.** log 653.	*Ans.* 6.482.
33. log 907.	*Ans.* 6.810.	**34.** log 4973.	*Ans.* 8.5118.
35. log 6481.	*Ans.* 8.7766.	**36.** log 9973.	*Ans.* 9.2076.

37. Find approximately the difference between $\log_{10}(x+1)$ and $\log_{10} x$ when x is large. *Ans.* $\dfrac{0.4343}{x}$.

38. Without using a table, find $\log_{10} 101$, $\log_{10} 999$, $\log_{10} 1002$, $\log_{10} 10003$.

39. For angles in the vicinity of 45°, find approximately the change (*a*) in $\log_{10} \sin\theta$, (*b*) in $\log_{10} \cos\theta$, (*c*) in $\log_{10} \tan\theta$, when θ changes by a small amount. *Ans.* (*a*) $0.43\,\Delta\theta$; (*b*) $-0.43\,\Delta\theta$; (*c*) $0.87\,\Delta\theta$.

40. Solve Ex. 39 for angles near 30°.

Ans. (*a*) $0.75\,\Delta\theta$; (*b*) $-0.25\,\Delta\theta$; (*c*) $1.003\,\Delta\theta$.

41. If x is a number and y its logarithm, $y = \log x$, find the uncertainty in x due to a small uncertainty in y. *Ans.* $e^y \Delta y$.

42. Solve Ex. 41 for common logarithms. *Ans.* $2.3026 \cdot 10^y \Delta y$.

43. If $\log_{10} x = 3.301 \pm 0.0005$, find x. *Ans.* 2000 ± 2.3.

61. Parametric equations. Instead of representing a curve by its ordinary cartesian equation, it is sometimes more convenient to determine the curve by two equations giving x and y in terms of some third variable. This variable is called a *parameter*, and the two equations are *parametric equations* of the curve.

For instance, the coördinates of a point moving in a plane are functions of the time:

$$x = \phi(t), \quad y = \psi(t).$$

These two equations may be considered as parametric equations of the path. Again, the equations of an ellipse in terms of the eccentric angle ϕ are

$$x = a\cos\phi, \quad y = b\sin\phi.$$

While it may be possible to eliminate the parameter, thus obtaining the ordinary cartesian equation of the curve, it is often more convenient not to do so.

When dealing with parametric equations, it is convenient to use differentials in finding derivatives, particularly the derivatives of higher order. The method is illustrated by the following

Example: Find y' and y'' when

$$x = t^2 + 1, \quad y = t^4 - 1.$$

Here

$$dx = 2\,t\,dt, \quad dy = 4\,t^3 dt,$$

so that

$$y' = \frac{dy}{dx} = 2\,t^2.$$

Again,

$$dy' = 4\,t\,dt, \quad dx = 2\,t\,dt,$$

and

$$y'' = \frac{dy'}{dx} = 2.$$

EXERCISES

Find y' and y'' in the following cases.

1. $x = 2\,t, \quad y = t^2.$ *Ans.* $y'' = \frac{1}{2}.$
2. $x = t^2, \quad y = 1 - t.$ *Ans.* $y'' = \dfrac{1}{4\,t^3}.$
3. $x = 1 + t^3, \quad y = t^2 - 1.$ *Ans.* $y'' = \dfrac{-2}{9\,t^4}.$
4. $x = 4\,t + 1, \quad y = t^3 - 2\,t^2.$ *Ans.* $y'' = \frac{3}{8}\,t - \frac{1}{4}.$
5. $x = u^2 - 2\,u, \quad y = u - 2.$ *Ans.* $y'' = \dfrac{-1}{4(u-1)^3}.$
6. $x = z^2 - 1, \quad y = z^2 - z.$ *Ans.* $y'' = \dfrac{1}{4\,z^3}.$
7. $x = a\cos 2\,\theta, \quad y = a\sin 2\,\theta.$ *Ans.* $y'' = -\dfrac{1}{a}\csc^3 2\,\theta.$
8. $x = a\cos^2\theta, \quad y = a\sin^2\theta.$ *Ans.* $y'' = 0.$
9. $x = a\cos^3\theta, \quad y = a\sin^3\theta.$ *Ans.* $y'' = \dfrac{1}{3\,a\cos^4\theta\sin\theta}.$
10. $x = a\cos\phi, \quad y = b\sin\phi.$ *Ans.* $y'' = \dfrac{-b}{a^2\sin^3\phi}.$
11. $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$ *Ans.* $y'' = \dfrac{-1}{a(1-\cos\theta)^2}.$
12. $x = \dfrac{2\,t}{1+t^2}, \quad y = \dfrac{1-t^2}{1+t^2}.$ *Ans.* $y'' = -\left(\dfrac{1+t^2}{1-t^2}\right)^3.$
13. $x = e^t, \quad y = 1 + e^{2t}.$ *Ans.* $y'' = 2.$
14. $x = \sinh u, \quad y = \cosh u.$ *Ans.* $y'' = \operatorname{sech}^3 u.$
15. $x = a\sec\phi, \quad y = b\tan\phi.$ *Ans.* $y'' = -\dfrac{b}{a^2}\cot^3\phi.$

Find the tangent to each of the following curves at the point indicated.

16. $x = 2t^2$, $y = t^2 - t$ at $t = 2$. *Ans.* $3x - 8y = 8$.

17. $x = t^3$, $y = 1 - t$ at $t = -1$. *Ans.* $x + 3y = 5$.

18. $x = a \sin\theta$, $y = a \cos^2\theta$ at $\theta = \frac{1}{6}\pi$. *Ans.* $4x + 4y = 5a$.

19. $x = a \tan\theta$, $y = a \cot\theta$ at $\theta = \frac{1}{4}\pi$. *Ans.* $x + y = 2a$.

20. $x = a \cos^3\theta$, $y = a \sin^3\theta$ at $\theta = \frac{3}{4}\pi$. *Ans.* $-\sqrt{2}\,x + \sqrt{2}\,y = a$.

CHAPTER VIII

CURVATURE

62. Differential of arc. Let s denote the length of the arc of the plane curve

$$y = f(x)$$

measured from some initial point P_0 to the point $P : (x, y)$, and suppose for definiteness that s increases as x increases. The arc s can be regarded as a function of x. Its derivative $\frac{ds}{dx}$ may be found as follows:

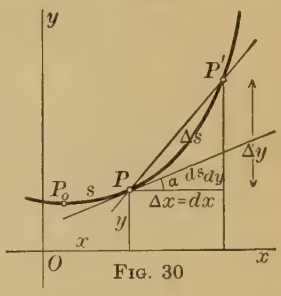


FIG. 30

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{\overline{PP'}} \cdot \frac{\overline{PP'}}{\Delta x} = \frac{\Delta s}{\overline{PP'}} \cdot \frac{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}}{\Delta x}$$

$$= \frac{\Delta s}{\overline{PP'}} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

where Δs is the length of the arc, $\overline{PP'}$ the length of the chord, from $P : (x, y)$ to $P' : (x + \Delta x, y + \Delta y)$. Since

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\overline{PP'}} = 1,$$

we have

$$\frac{ds}{dx} = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{1}$$

If s decreases as x increases, then

$$\frac{\Delta s}{\Delta x} = -\frac{\Delta s}{\overline{PP'}} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

and

$$\frac{ds}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2}$$

After squaring and clearing of fractions, equation (1) (or (2)) becomes

$$\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2;$$

i.e. ds is the hypotenuse of the right triangle whose sides are dx and dy.

If the tangent to the curve at P makes an angle α with Ox, then

$$\cos\alpha = \frac{dx}{ds}, \quad \sin\alpha = \frac{dy}{ds}.$$

63. Curvature. We say in ordinary language that a curve whose direction changes rapidly has great *curvature*, or is sharply curved. Thus a circular arc is said to have greater curvature when the radius is small than when it is large. This somewhat vague idea may be made precise as follows.

Consider, first, two points P, P' on a circle, and denote the arc PP' by Δs, the angle between the tangents at P, P' by $\Delta\alpha$. The quotient $\frac{\Delta\alpha}{\Delta s}$ is evidently the change in the direction of the curve, per unit of arc. It is evident that the central angle subtended by Δs is equal to $\Delta\alpha$; hence

$$\Delta s = r\Delta\alpha,$$

where r is the radius, and

$$\frac{\Delta\alpha}{\Delta s} = \frac{1}{r}.$$

That is, in the case of the circle the quotient $\frac{\Delta\alpha}{\Delta s}$ is constant; it is called the *curvature* of the circle.

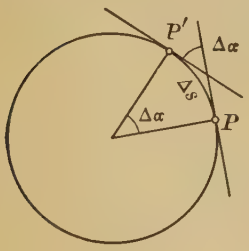


FIG. 31

If now the curve in question is not a circle, the direction of the curve no longer changes uniformly, and the quotient $\frac{\Delta\alpha}{\Delta s}$ represents merely the *average curvature* of the arc Δs. But if P' be made to approach P along the curve, so that

Δs and $\Delta\alpha$ approach 0, the quantity $\frac{\Delta\alpha}{\Delta s}$ in general approaches a limit $\frac{d\alpha}{ds}$, which is called the *curvature at the point* P:

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}.$$

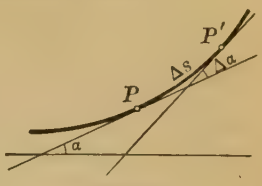


FIG. 32

The definition is of course independent of the particular coördinate system used; the angle α is the angle made by the tangent at P with *any* fixed line in the plane of the curve. When the equation of the curve is given in cartesian coördinates, it is convenient to take α as the slope-angle of the tangent — *i.e.* the angle between the tangent and the x-axis. The curvature κ is then easily expressed in terms of the coördinates. For,

$$\tan\alpha = \frac{dy}{dx} = y',$$

$$\alpha = \arctan y',$$

$$d\alpha = \frac{dy'}{1+y'^2} = \frac{y''\,dx}{1+y'^2}.$$

Also, by § 62,

$$ds = \sqrt{1+y'^2}\,dx.$$

Hence

(1) $$\kappa = \frac{d\alpha}{ds} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}.$$

It is customary to consider κ as essentially positive, so that, strictly speaking, we should write

$$\kappa = \left|\frac{d\alpha}{ds}\right| = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}},$$

where the symbol $|a|$ means the absolute or numerical value of a.

When $y' = 0$, formula (1) reduces to

$$\kappa = y''.$$

Thus the value of the second derivative at any point is equal to the curvature at that point when the coördinate axes are so chosen that the first derivative is 0.

64. Radius of curvature. The reciprocal of the curvature is called the *radius of curvature*, and is denoted by ρ:

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\alpha} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}.$$

This quantity is also to be considered as essentially positive.

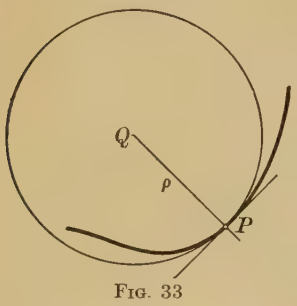


FIG. 33

If a length equal to the radius of curvature ρ at the point P be laid off on the normal from P toward the concave side of the curve, the extremity Q of this segment is called the *center of curvature*. It can be shown that the circle with radius ρ and center Q represents the curve near P more closely than any other circle. This circle is called the *osculating circle*, or *circle of curvature*.

In general, the circle of curvature crosses the curve at P, as in Fig. 33.

EXERCISES

Find the radius of curvature of the following curves.

1. $y = x^2$ (*a*) at any point; (*b*) at (2, 4). *Ans.* (*b*) $\frac{17}{2}\sqrt{17}$.
2. $y = x^3 - 6x$ at $(1, -5)$. *Ans.* $\frac{5}{3}\sqrt{10}$.
3. $y = x^3 + x^2 - x + 2$ at $x = 0$. *Ans.* $\sqrt{2}$.
4. $y = x^4 - 3x^3$ at $x = 1$. *Ans.* $\frac{13}{3}\sqrt{26}$.
5. The parabola $y^2 = 4ax$ at any point. *Ans.* $\frac{(4a^2 + y^2)^{\frac{3}{2}}}{4a^2}$.
6. The hyperbola $2xy = a^2$. *Ans.* $\frac{(x^2 + y^2)^{\frac{3}{2}}}{a^2}$.
7. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $\frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.
8. The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $3(axy)^{\frac{1}{3}}$.
9. The curve $y = \tan x$ at $x = \frac{1}{4}\pi$. *Ans.* $\frac{5}{4}\sqrt{5}$.
10. The curve $y = \log \cos x$. *Ans.* $\sec x$.
11. The curve $y = \log \cosh x$. *Ans.* $\cosh^2 x(1 + \tanh^2 x)^{\frac{3}{2}}$.
12. The curve $x = 3t^2$, $y = 2t^3$. *Ans.* $6t(1 + t^2)^{\frac{3}{2}}$.

13. The curve $x = 2t + 3, \quad y = t^2 - 2t.$ *Ans.* $2(t^2 - 2t + 2)^{\frac{3}{2}}$.

14. The curve $x = 3t^2, \quad y = 3t - t^3.$ *Ans.* $\frac{3}{2}(1 + t^2)^2$.

15. The ellipse $x = a\cos\phi, \quad y = b\sin\phi.$

16. The hyperbola $x = a\tan\phi, \quad y = a\cot\phi.$

Ans. $\frac{1}{2}a\tan^3\phi\,(1 + \cot^4\phi)^{\frac{3}{2}}$.

17. The curve $x = a\cos^3\theta, \quad y = a\sin^3\theta.$ *Ans.* $3a\cos\theta\sin\theta$.

18. The curve $x = a\cos^4\theta, \quad y = a\sin^4\theta.$

Ans. $2a\cos^6\theta\,(1 + \tan^4\theta)^{\frac{3}{2}}$.

19. Show that the radius of curvature of a circle is the radius of the circle.

20. Show that the curvature of a straight line is everywhere 0.

21. Show that the curvature at a point of inflection is 0.

Find the points of maximum curvature on the following curves.

22. A parabola. *Ans.* The vertex.

23. The curve $y = x^3$. *Ans.* $x = \pm(\frac{1}{45})^{\frac{1}{4}}$.

24. The curve $y = x^4$. *Ans.* $x = \pm(\frac{1}{56})^{\frac{1}{6}}$.

25. The curve $y = \sin x$. *Ans.* $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \cdots$.

26. The curve $y = \log\sin x$.

27. The catenary $y = a\cosh\dfrac{x}{a}$. *Ans.* $(0, a)$.

28. The curve $y = \sinh x$. *Ans.** $x = \pm\log(1 + \sqrt{2})$.

29. The curve $y = e^x$. *Ans.* $(-0.347, 0.707)$.

30. The curve $y = \log x$. *Ans.* $x = \frac{1}{2}\sqrt{2}$.

31. The curve $y = x + \log x$. *Ans.* $x = -\frac{1}{8} + \frac{1}{8}\sqrt{17}$.

* See Ex. 44, p. 74.

CHAPTER IX

TIME-RATES. MOTION

65. Miscellaneous applications. The fact that the derivative of a function is identical with its rate of change leads to a variety of applications; those in which time is the independent variable are especially important.

If in any problem the quantity whose rate of change is to be found can be expressed directly as a function of time, the result can of course be obtained at once by differentiating with respect to the time. Frequently, however, the problem is best solved by expressing the quantity in question in terms of another quantity whose rate of change is known, and then differentiating the equation connecting them. The method may be explained by an

Example: Water is flowing into a conical reservoir 20 ft. deep and 10 ft. across the top, at the rate of 15 cu. ft. per minute. Find how fast the surface is rising when the water is 8 ft. deep.

The volume of water is

$$V = \tfrac{1}{3}\pi r^2 h.$$

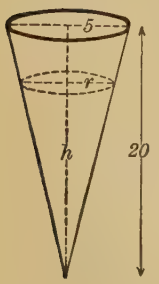


FIG. 34

By similar triangles,

$$\frac{r}{h} = \frac{5}{20}, \quad r = \frac{1}{4}h.$$

Hence

$$V = \frac{\pi h^3}{48},$$

$$dV = \frac{\pi h^2}{16}\,dh,$$

$$\frac{dV}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}.$$

But we have given that

$$\frac{dV}{dt} = 15,$$

so that

$$\frac{\pi h^2}{16}\frac{dh}{dt} = 15, \quad \frac{dh}{dt} = \frac{240}{\pi h^2}.$$

When $h = 8$,

$$\frac{dh}{dt} = \frac{15}{4\pi} = 1.19 \text{ ft. per minute.}$$

EXERCISES

1. Water is flowing into a cylindrical tank at the rate of 10 cu. ft. per sec. If the radius of the tank is 3 ft., how fast is the surface rising?

Ans. 0.35 ft. per sec.

2. The cross-section of a tank is a square of side 2 ft. If water flows in at the rate of 100 cu. in. per sec., how fast does the surface rise? *Ans.* 0.17 in. per sec.

3. A rectangular trough is 10 ft. long and 1 ft. wide. Find how fast the surface rises if water flows in at the rate of 2 cu. ft. per min.

Ans. 0.2 ft. per min.

4. A triangular trough is 10 ft. long, 2 ft. across the top, and 2 ft. deep. If water flows in at the rate of 6 cu. ft. per min., find how fast the surface is rising when the water is 6 in. deep. *Ans.* 1.2 ft. per min.

5. A man 6 ft. tall walks away from a lamp-post 12 ft. high at the rate of 3 mi. per hr. How fast does the end of his shadow move?

6. In Ex. 5, how fast does the shadow lengthen?

7. In the example of § 65, find how fast the water is flowing in if, when the water is 5 ft. deep, the surface is rising 2 ft. per min.

Ans. 9.8 cu. ft. per min.

8. The base of a right triangle is 6 ft. If the altitude increases at the rate of 2 ft. per min., find the rate of change of the area and of the perimeter when the altitude is 4 ft.

Ans. $\frac{dA}{dt} = 6$ sq. ft. per min.; $\frac{dP}{dt} = 3.11$ ft. per min.

9. The altitude of a right circular cone is 30 in. If the radius changes at the rate of 2 in. per sec., find the rate of change of the volume when the radius is 3 in. *Ans.* $120\,\pi$ cu. in. per sec.

10. In Ex. 9, find the rate of change of the angle at the vertex.

Ans. $\frac{40}{303}$ rad. per sec.

11. In Ex. 8, find the rate of change of the angle at the base.

Ans. $\frac{3}{13}$ rad. per min.

12. The altitude of a right pyramid is 8 ft.; the base is a square of side x. If $\frac{dx}{dt} = 3$ ft. per min., find $\frac{dV}{dt}$ when $x = 2$ ft.

Ans. 32 cu. ft. per min.

13. Two trains start from the same point at the same time, one going east at the rate of 40 mi. per hr., the other north 60 mi. per hr. At what rate do they separate? *Ans.* 72.1 mi. per hr.

14. Solve Ex. 13 if the second train goes N. 30° E.

Ans. 52.9 mi. per hr.

15. A ladder 10 ft. long leans against a wall. If the lower end moves outward at the rate of 18 in. per sec., find how fast the top is descending when the lower end is 6 ft. from the wall. *Ans.* $\frac{9}{8}$ ft. per sec.

16. In Ex. 15, find how fast the slope of the ladder is decreasing.

Ans. $\frac{25}{48}$ per sec.

17. In Ex. 15, find the rate at which the angle between the ladder and the floor is decreasing. *Ans.* $\frac{3}{16}$ rad. per sec.

18. A ship sails north 2 miles, then turns east. If it moves at the rate of 10 mi. per hr., how fast is it leaving the starting point after sailing 30 min.? *Ans.* $\frac{30}{13}\sqrt{13}$ mi. per hr.

19. In Ex. 18, if a searchlight placed at the point of departure follows the ship, how fast is the light rotating at the end of 30 minutes?

Ans. 1° 28′ per min.

20. A trapezoidal trough is 12 ft. long, 3 ft. wide at the top, 2 ft. wide at the bottom, and 2 ft. deep. If water flows in at the rate of 20 cu. ft. per min., find how fast the surface is rising when the water is 1 ft. deep. *Ans.* $\frac{2}{3}$ ft. per min.

21. A light stands 30 ft. from a house, and 20 ft. from the path leading from the house to the street. A man walks along the path at 5 ft. per sec. How fast does his shadow move on the wall when he is 20 ft. from the house? *Ans.* 30 ft. per sec.

22. A man on a wharf 20 ft. above the water pulls in a rope, to which a boat is attached, at the rate of 4 ft. per sec. At what rate is the boat approaching the shore when there is 25 ft. of rope out?

Ans. $6\frac{2}{3}$ ft. per sec.

23. A kite is 120 ft. high, with 130 ft. of cord out. If the kite moves horizontally 4 mi. per hr. directly away from the boy flying it, how fast is the cord being paid out? *Ans.* $\frac{20}{13}$ mi. per hr.

24. In Ex. 23, find the rate of change of the angle at which the cord is inclined to the vertical. *Ans.* $\frac{176}{4225}$ rad. per sec.

25. Two railroad tracks intersect at right angles. At noon there is a train on each track approaching the crossing at 40 mi. per hr., one being 100 mi., the other 200 mi. distant. Find (*a*) how fast they are approaching each other, (*b*) when they will be the nearest together, and (*c*) what will be their minimum distance apart.

Ans. (*b*) 3 : 45 P.M.; (*c*) 70.7 mi.

26. A light is placed on the ground 30 ft. from a building. A man 6 ft. tall walks from the light toward the building, at the rate of 5 ft. per sec. Find the rate at which his shadow on the wall is shortening when he is 15 ft. from the building. *Ans.* 4 ft. per sec.

27. Solve Ex. 26 if the light is 10 ft. above the ground.

Ans. $\frac{5}{8}$ ft. per sec.

28. The city of A is 10 mi. north of B. A car starts east from A at 30 mi. per hr.; at the same time another car starts east from B at 60 mi. per hr. How fast are the cars separating after 10 minutes?

Ans. $6\sqrt{5}$ mi. per hr.

29. A ship sails north 20 miles and then turns E. 30° N. If the ship's speed is 10 mi. per hr., find how fast its distance from the starting point is increasing at the end of 4 hours. *Ans.* 8.7 mi. per hr.

30. As a man walks across a bridge at the rate of 5 ft. per sec. a boat passes directly beneath him at 10 ft. per sec. If the bridge is 30 ft. above the water, how fast are the man and the boat separating 3 seconds later? *Ans.* $8\frac{1}{3}$ ft. per sec.

31. An elevated train on a track 30 ft. above the ground crosses a street at the rate of 20 ft. per sec. at the instant that an automobile, approaching at the rate of 30 ft. per sec., is 40 ft. up the street. Find how fast the train and the automobile are separating 2 seconds later.

Ans. 26.0 ft. per sec.

32. In Ex. 31, find when the train and the automobile are nearest together. *Ans.* $\frac{12}{13}$ sec.

33. An arc light hangs at a height of 30 ft. above the center of a street 60 ft. wide. A man 6 ft. tall walks along the sidewalk at the rate of 4 ft. per sec. How fast is his shadow lengthening when he is 40 ft. up the street? *Ans.* 0.8 ft. per sec.

34. In Ex. 33, how fast is the tip of the shadow moving?

Ans. 5 ft. per sec.

66. Velocity and acceleration in rectilinear motion. If a point P moving in a straight line describes equal distances in equal times, its motion is said to be *uniform*. The space passed over per unit time is called the *velocity* of the moving point.

If the motion is not uniform, we introduce the idea of *velocity at a point*, or *instant*. Choose an arbitrary origin O in the line of motion (not necessarily the starting point) and let x denote the distance from O to P at the time t.

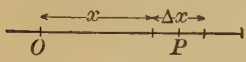


FIG. 35

If a distance Δx is described during the time-interval Δt, the quotient $\frac{\Delta x}{\Delta t}$ is the *average velocity* during that interval. The limit of this quotient as Δt approaches 0 is called the *velocity at the point* P:

$$v = \frac{dx}{dt}.$$

Thus the velocity is the time-rate of change of space described (cf. § 5).

When v changes sign, the sense of the motion is reversed, and x passes through a maximum or minimum. This of course agrees with the theory of § 26.

In rectilinear motion, the rate of change of velocity is called *acceleration:*

$$j = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If the acceleration j is constant, the motion is said to be *uniformly accelerated.*

An important case of uniformly accelerated motion is that of a body falling toward the earth from a point near the earth's surface, all resistances being neglected. The attraction of the earth gives the body an acceleration g, called the *acceleration of gravity*, equal to 32 ft. per second per second approximately.

It follows from the above definitions that when x is given as a function of t, the velocity and acceleration can be found directly, and the nature of the motion can be completely determined.

In any problem, it is instructive to draw the graphs of x, v, and j as functions of t. In doing this, it should be remembered that the graph of v is the first derived curve (§ 32),

the graph of j is the second derived curve, corresponding to the graph of x.

Example: A body moves in a straight line according to the law

$$x = t^3 - 9t^2 + 15t,$$

where x is measured in feet and t in seconds. Discuss the motion.

Differentiation gives

$$v = 3t^2 - 18t + 15,$$
$$j = 6t - 18.$$

When $t = 0$, we find $x = 0$, $v = 15$, $j = -18$; the body therefore starts at the origin, with an initial velocity of 15 ft. per sec. in the positive direction, but with a negative acceleration of 18 ft. per sec. per sec. The velocity is 0 when $t = 1$ or 5, the corresponding values of x being 7 and -25. The body therefore moves out to A (Fig. 36), where $OA = 7$, returns to B ($OB = -25$), and then moves indefinitely in the positive direction with continually increasing velocity. When $j = 0$, $t = 3$, $v = -12$, $x = -9$; thus the greatest negative velocity is -12 ft. per sec., occurring at C.

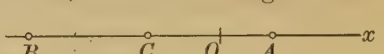


FIG. 36

EXERCISES

In each of the following cases, a body moves in a straight line according to the given law. Find the initial values of x, v, and j, examine x and v for maxima and minima, and describe the motion fully.

1. $x = 16t^2 - 4t$.
2. $x = 12t - 16t^2$.
3. $x = 4t^2 + 8t - 2$.
4. $x = \frac{1}{2}t^2 + 6t + 2$.
5. $x = 2t^3 - 6t$.
6. $x = t^3 - 6t^2 + 9t$.
7. $x = 2 - 12t + 9t^2 - 2t^3$.
8. $x = \frac{1}{4}t^3 - \frac{3}{2}t^2$.
9. $x = t^3 - 3t^2 + 3t$.
10. $x = (1 - 2t)^3$.
11. $x = \frac{1}{2}t^4 - \frac{5}{3}t^3 + 2t^2 - t$.
12. $x = t^4 - 8t^3 + 22t^2 - 24t$.
13. $x = 2 \sin \frac{1}{2}t$.
14. $x = \frac{1}{3} \cos 2t$.
15. $x = -3 \cos \frac{1}{3}t$.
16. $x = 1 - \sin 3t$.
17. $x = t - \sin t$.
18. $x = t + \cos 2t$.
19. $x = 1 - e^{-t}$.
20. $x = 3 + e^{-2t}$.

21. $x = 2(e^{-2t} - e^{-3t})$.
22. $x = e^{-t} - e^{-3t}$.
23. $x = e^{t} + 2\,e^{-t}$.
24. $x = bte^{-kt}$.
25. $x = t - 4\,e^{-t}$.
26. $x = bt^{2}e^{-kt}$.
27. $x = \log\,(1 + 2\,t)$.
28. $x = \log(1 + 2\,t) - t$.
29. $x = 16\,t^{2} - 24\log(1 + t)$.
30. $x = t^{2} - \frac{9}{2}\,t + 5\log\,(1 + t)$.
31. $x = 2\cos t - \cos 2\,t - 1$.
32. $x = 2\sin t + \cos 2\,t$.
33. $x = e^{-t}\sin t$.
34. $x = e^{-3t}\cos \frac{1}{2}\,t$.
35. $x = e^{-t}(\sin t + \cos t)$.
36. $x = e^{t}\sin t$.

67. Vectors. A straight line segment of definite *length*, *direction*, and *sense* is called a *vector*. Vectors are of great importance in physics because they can be used to represent velocities, accelerations, forces, and other fundamental quantities.

The *resultant* of two vectors AB, AC is the diagonal AD of the parallelogram having AB, AC as adjacent sides. Two forces acting on the same particle are equivalent to a single force, their resultant; similarly for other vectors. This is the *parallelogram law*. The operation of finding the resultant by the parallelogram law is called *geometric addition*, or *vector addition*.

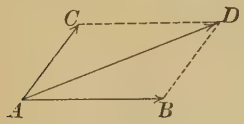


FIG. 37

The original vectors AB, AC are called *components* of AD. It is evident that any vector may be resolved into components in an infinite number of ways.

68. Velocity in curvilinear motion. If a moving point describes a plane curve, its coördinates are functions of the time:

$$x = \phi(t),\ y = \psi(t).$$

The distance s passed over along the curve is also a function of the time.

The velocity at any point P is defined as the vector, laid off from P on the tangent to the path, of magnitude

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

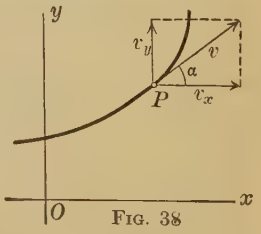


FIG. 38

The components of the velocity parallel to the coördinate axes are

$$v_x = v \cos \alpha, \quad v_y = v \sin \alpha,$$

where α is the angle between Ox and the tangent at P. By § 62,

$$v \cos \alpha = \frac{ds}{dt} \cdot \frac{dx}{ds} = \frac{dx}{dt}, \quad v \sin \alpha = \frac{ds}{dt} \cdot \frac{dy}{ds} = \frac{dy}{dt},$$

so that

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}.$$

Thus the rectangular components of the velocity of P are the velocities of the projections P_x and P_y of P on the coördinate axes.

By § 67, the total velocity v is

$$v = \sqrt{v_x^2 + v_y^2},$$

inclined to the x-axis at an angle

$$\alpha = \arctan \frac{v_y}{v_x}.$$

The equations

$$x = \phi(t), \; y = \psi(t)$$

may be regarded as *parametric equations of the path* (§ 61) in terms of the parameter t. The cartesian equation may be obtained by eliminating the parameter.

69. Acceleration in curvilinear motion. Suppose the velocity of the moving point P at the time t is v, at the time $t + \Delta t$ is $v' = v + \Delta v$, where Δv is the *vector* which, geometrically added to v, produces v'. If v and v' be laid off from a common origin O, the third side VV' of the triangle (Fig. 40) is evidently Δv. Now as Δt approaches 0, Δv does likewise;

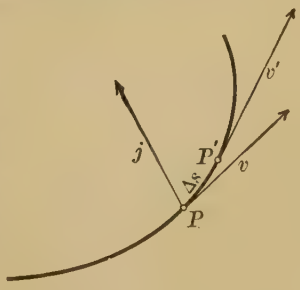


Fig. 39

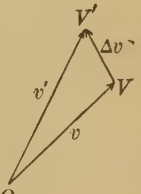


Fig. 40

but in general the ratio $\frac{\Delta v}{\Delta t}$ approaches a definite limit, and the direction of Δv approaches a definite limiting direction.

The vector of length

$$j = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t},$$

laid off in the limiting direction of Δv, is called the *acceleration* of P at the time t. It is the so-called *geometric derivative*, or *vector derivative*, of v with respect to t.

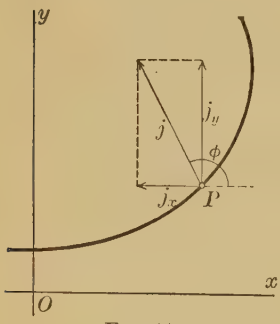


FIG. 41

To find an expression for j in terms of the coördinates of P, we may resolve j into components j_x and j_y parallel to the coördinate axes. Denoting by ϕ' the angle between Δv and the x-axis, let us project the triangle OVV' (Fig. 40) on Ox:

$$\Delta v \cos \phi' = v_x' - v_x = \Delta v_x.$$

Dividing by Δt, we get

$$\frac{\Delta v}{\Delta t} \cos \phi' = \frac{\Delta v_x}{\Delta t},$$

whence, in the limit when Δt approaches 0,

$$j \cos \phi = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},$$

where ϕ, the limiting value of ϕ', is the angle between j and the x-axis. Similarly

$$j \sin \phi = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

Thus

$$\boldsymbol{j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}},$$

$$\boldsymbol{j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}}.$$

The total acceleration j is

$$j = \sqrt{j_x^2 + j_y^2},$$

inclined to the x-axis at an angle

$$\phi = \arctan \frac{j_y}{j_x}.$$

70. Rotation. If a point moves in a circle at a uniform rate, so that the radius vector sweeps out equal angles in equal times, the angle swept out per unit time is called the *angular velocity*. If the motion is not uniform, the *angular velocity at a particular instant* is defined as

$$\omega = \frac{d\theta}{dt};$$

i.e. the angular velocity ω is the rate of change of the angle θ described by the radius vector.

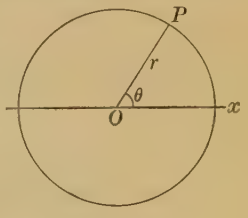


FIG. 42

Since $ds = r\,d\theta$, where r is the radius of the circle, it follows that the linear velocity $v = \dfrac{ds}{dt}$ and the angular velocity ω are connected by the relation

$$v = \omega r.$$

The rate of change of the angular velocity is called the *angular acceleration*, and is denoted by α:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

EXERCISES

1. A man can row a boat 5 mi. per hr. If he pulls at right angles to the course of a river 2 mi. wide having a current of 3 mi. per hr., where and when will he reach the opposite shore?

2. In Ex. 1, if the man wishes to land directly opposite his starting point, in what direction must he row and how long will it take him to cross? *Ans.* 30 min.

3. A steamship is moving at the rate of 12 mi. per hr. A man walks across the deck at right angles to the ship's course, at the rate of 5 mi. per hr. If the deck is 40 ft. wide, how far is he finally from his starting point?

A point moves in a plane curve, its coördinates being determined by the given formulas. Find the cartesian components of velocity and acceleration; also obtain the cartesian equation of the path, and trace the curve. Describe the motion fully.

4. $x = 2t,\ y = 4t^2 - 2.$

5. $x = 8t,\ y = 8t - 16t^2.$

6. $x = a\cos kt,\ y = a\sin kt.$

7. $x = 2\cos t,\ y = \sin t.$

8. $x = \frac{1}{3}\sin 3t,\ y = \frac{1}{3}\cos 3t.$

9. $x = \cos 2t,\ y = 8 - 8\sin 2t.$

10. $x = 2t,\ y = 3\sin t.$

11. $x = \sin 3t,\ y = \frac{1}{3}t.$

12. $x = e^{-2t},\ y = t - e^{-t}.$

13. $x = t,\ y = e^{-t}\sin 2t.$

14. A point describes the parabola $y^2 = 4x$ with a constant velocity of 6 ft. per sec. Find v_x, v_y, j_x, and j_y at the point (1, 2).

Ans. $j_x = \frac{9}{2}, j_y = -\frac{9}{2}$.

15. A point describes the parabola $y = x^2 + 2x + 3$. If $v_x = 3$, find v_y, j_x, and j_y at any point. *Ans.* $j_y = 18$.

16. A flywheel 2 ft. in diameter makes 100 revolutions per min. Find its angular velocity in radians per sec., and the linear velocity of a point on the rim. What constant angular retardation (negative acceleration) would bring it to rest in 10 sec.?

17. A point moves in a circle in such a way that

$$\theta = 4t^2 - 3t.$$

Find ω and α, and draw the graphs of θ, ω, and α as functions of t.

18. In Ex. 17, find v_x and v_y when $t = 1$ if the radius of the circle is 10 ft. *Ans.* $v_x = -42.1$, $v_y = 27.0$ ft. per sec.

CHAPTER X

INDETERMINATE FORMS

71. The Law of the Mean. Let $f(x)$ be a function which is continuous, one-valued, and differentiable in the interval $x = a$ to $x = b$. It is geometrically obvious that at some point such as P (Fig. 43) the tangent must be parallel to the secant SQ. Now the slope of the secant is

$$\frac{RQ}{SR} = \frac{f(b) - f(a)}{b - a};$$

the slope of the tangent at P is $f'(x_1)$, where x_1 is the abscissa of P. Hence

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

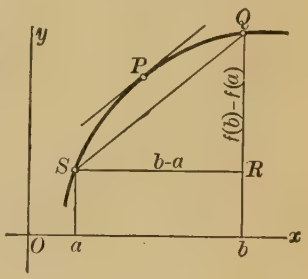


FIG. 43

or

$$f(b) - f(a) = (b - a)f'(x_1), \qquad a < x_1 < b. \tag{1}$$

This formula is called the *Law of the Mean.*

Placing

$$a = x, \qquad b = x + \Delta x,$$

and denoting by θ a positive number less than unity, we obtain a useful alternative form of the Law of the Mean:

$$f(x + \Delta x) = f(x) + \Delta x f'(x + \theta \Delta x), \qquad 0 < \theta < 1.$$

EXERCISES

1. At what point of the parabola $y = x^2$ is the tangent parallel to the secant through the points $x = 0$, $x = 2$?

2. At what point on the curve $y = x^3 - x$ is the tangent parallel to the secant through (1, 0) and (2, 6)? Draw the figure.

3. Find the point on the curve $y = \log x$ where the tangent is parallel to the secant through the points $x = 1$, $x = 2$. *Ans.* $x = \dfrac{1}{\log 2}$.

4. Find the point on the curve $y = e^x$ where the tangent is parallel to the secant through the points $x = 0$, $x = 1$.

5. Show that there are two points on the curve $y = \frac{1}{3}x^3 - x$, in the interval $x = -1$ to $x = 1$, where the tangent is parallel to the secant joining the endpoints. Draw the curve.

Ans. $(\pm \frac{1}{3}\sqrt{3}, \mp \frac{2}{27}\sqrt{3})$.

Show that the Law of the Mean does not hold in the following cases, and give the reason.

6. Curve $xy = 1$, interval $x = -1$ to $x = 1$.

7. Curve $y = x^{\frac{2}{3}}$, interval $x = -1$ to $x = 1$.

8. Curve $y = \log x$, interval $x = 0$ to $x = 1$.

9. Curve $y = \tan x$, interval $x = 0$ to $x = \pi$.

72. The indeterminate forms $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$. If two functions $f(x)$, $F(x)$ both vanish at $x = a$:

$$f(a) = 0, \quad F(a) = 0,$$

their quotient $\dfrac{f(x)}{F(x)}$ assumes the "indeterminate form" $\dfrac{0}{0}$ at $x = a$, and is undefined at that point. Nevertheless the *limit* $\lim\limits_{x \to a} \dfrac{f(x)}{F(x)}$ may exist. This fact is illustrated in the derivation of the fundamental differentiation formulas, where in each case both numerator and denominator of the difference quotient $\dfrac{\Delta y}{\Delta x}$ approach 0, yet the derivative, which is the limit of that quotient, exists.

In case the fraction $\dfrac{f(x)}{F(x)}$ does approach a limit when x approaches a, we lay down the following *definition:*

$$\left.\frac{f(x)}{F(x)}\right]_{x=a} = \lim_{x \to a} \frac{f(x)}{F(x)}.$$

The function $\frac{f(x)}{F(x)}$ thus becomes continuous at the point $x = a$, by the definition of continuity (§ 9).

It may be possible to evaluate the limit by means of more or less obvious transformations of $\frac{f(x)}{F(x)}$, as was done in deriving the differentiation formulas. In many cases the limit may be obtained by a method that will now be developed.

In formula (1) of § 71, let us take $b = x$. This gives

$$f(x) = f(a) + (x - a)f'(x_1),$$
$$F(x) = F(a) + (x - a)F'(x_2),$$

where x_1 and x_2 lie between a and x. But by hypothesis

$$f(a) = F(a) = 0.$$

Hence

$$\frac{f(x)}{F(x)} = \frac{(x - a)f'(x_1)}{(x - a)F'(x_2)} = \frac{f'(x_1)}{F'(x_2)}.$$

As x approaches a, x_1 and x_2 must do likewise, and we have, by Theorem III of § 8,

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x_1)}{F'(x_2)} = \lim_{x \to a} \frac{f'(x)}{F'(x)} = \frac{f'(a)}{F'(a)},$$

provided $f'(a)$ and $F'(a)$ exist and $F'(a) \neq 0$.

If $f(x)$ and $F(x)$ both increase indefinitely as x approaches a:

$$\lim_{x \to a} f(x) = \infty, \qquad \lim_{x \to a} F(x) = \infty,$$

the fraction $\frac{f(x)}{F(x)}$ is said to assume the indeterminate form $\frac{\infty}{\infty}$ at $x = a$. Here again it may happen that $\lim_{x \to a} \frac{f(x)}{F(x)}$ exists, and it can be shown that the same method may be applied in this case as in the case just treated.

It may happen that the fraction $\frac{f'(x)}{F'(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. In this case we may differentiate numerator and denominator again, and repeat as many times as necessary.

Finally, we may have $a = \infty$: *i.e.* $\frac{f(x)}{F(x)}$ approaches the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when x increases indefinitely. In this case the rule holds also.

The results of this article may be summarized in the following

THEOREM: *If the fraction* $\frac{f(x)}{F(x)}$ *assumes the indeterminate form* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ *when* $x = a$, *then*

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)},$$

provided the latter limit exists.

Thus we may differentiate the numerator and the denominator *separately*, and take the limit of the new fraction thus formed. It must be borne clearly in mind, however, that the theorem applies only to *fractions* in which the numerator and the denominator *both approach* 0 *or both increase indefinitely.*

Example: Evaluate $\lim_{x \to 0} \frac{\tan x}{x}$.

This fraction takes the form $\frac{0}{0}$ when $x = 0$, so that the theorem applies:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1.$$

73. The indeterminate form $0 \cdot \infty$. Given the product of two functions $f(x) \cdot F(x)$, suppose that as x approaches a one function approaches 0 while the other increases indefinitely. The product is then said to take the indeterminate form $0 \cdot \infty$.

If we write

$$f(x) \cdot F(x) = \frac{f(x)}{\frac{1}{F(x)}},$$

it appears that the quotient last written assumes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the theorem of § 72 may be applied.

Example: Evaluate $\lim_{x \to 0^+} x \log x$.

This takes the form $0 \cdot \infty$. If we write it in the equivalent form $\dfrac{\log x}{\dfrac{1}{x}}$, the theorem of § 72 may be used:

$$\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0.$$

EXERCISES

Evaluate the following limits, using the theorem of § 72 when applicable.

1. $\lim_{x \to \frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}}$.

2. $\lim_{x \to \infty} \dfrac{x^2}{e^x}$.

3. $\lim_{x \to 4} \dfrac{x^2 - 16}{x^2 + x - 20}$.

4. $\lim_{x \to -1} \dfrac{\log(x+2)}{x+1}$.

5. $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$.

6. $\lim_{x \to 0} \dfrac{\log \cos x}{x}$.

7. $\lim_{x \to \infty} \dfrac{3x^2 - 4x}{2x^2 - 3x + 1}$.

8. $\lim_{x \to \infty} \dfrac{x - 1}{x^2 + 1}$.

9. $\lim_{x \to 0^+} x \log \sin x$.

10. $\lim_{x \to \infty} xe^{-x}$.

11. $\lim_{n \to \infty} \dfrac{n}{2^n}$. *Ans.* 0.

12. $\lim_{\phi \to \pi} \sin \phi \cot \phi$. *Ans.* -1.

13. $\lim_{x \to \infty} \dfrac{\cos x}{x}$. *Ans.* 0.

14. $\lim_{x \to \infty} (x^2 - x)$.

15. $\lim_{x \to \infty} \dfrac{\log(1 + e^x)}{x}$.

16. $\lim_{n \to \infty} \dfrac{3^n}{\pi^n}$. *Ans.* 0.

17. $\lim_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$.

18. $\lim_{x \to 0} x \sin \dfrac{1}{x}$. *Ans.* 0.

19. $\lim_{x \to \infty} \dfrac{\sin x}{\cos 2x}$.

20. $\lim_{\theta \to \frac{\pi}{2}} \dfrac{\tan \theta}{\tan 3\theta}$.

21. $\lim_{x \to a} \dfrac{\sin x - \sin a}{x - a}$.

22. $\lim_{x \to 1} \dfrac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$.

23. $\lim_{x \to \infty} \dfrac{\tan x}{x}$. *Ans.* No limit.

24. $\lim_{x \to \infty} \dfrac{x}{\cos x}$. *Ans.* No limit.

CHAPTER XI

CURVE TRACING

74. Horizontal and vertical asymptotes. In Chapter IV we developed certain results in the theory of curves, and employed those results to trace the graphs of polynomials. In the present chapter we shall attempt a more general treatment of the subject of curve tracing.

As the point of contact of a tangent to a curve recedes indefinitely from the origin, the tangent may or may not approach a limiting position. If it does, the line approached is called an *asymptote*.

For example, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has the lines

$$\frac{x}{a} \pm \frac{y}{b} = 0$$

as asymptotes. On the other hand, the parabola has no asymptotes, since as the point of tangency recedes the tangent does not approach any limiting position.

Although general methods for finding asymptotes exist, they are frequently difficult to apply. Asymptotes parallel to the coördinate axes can usually be determined by the following rule * (cf. §§ 10–11):

In general, if y becomes infinite as x approaches a definite limit a, the line $x = a$ is an asymptote; if y approaches b as x becomes infinite, the line $y = b$ is an asymptote.

For the sake of brevity, the problem of finding asymptotes oblique to the axes will be omitted.

* In rare instances, the derivative may behave in such a way that although the conditions of the rule are satisfied, the tangent does not approach any limiting position, and hence there is no asymptote; but the rule holds in most cases that are apt to arise in practice.

Example: Find the horizontal and vertical asymptotes of the curve

$$y^2 = \frac{x^2}{x^2 - 1}. \tag{1}$$

When x becomes infinite, the right member of (1) approaches 1: since y^2 approaches 1, y approaches ± 1, and the lines $y = \pm 1$ are asymptotes.

When x approaches ± 1, the right member of (1) becomes infinite; the lines $x = \pm 1$ are asymptotes.

75. Restriction of the curve to definite regions. It is frequently possible to show that the curve is confined to certain definite portions of the plane, and a result of this kind is of great value in tracing the curve. While no general directions can be given, it is often useful to solve the equation for y (or y^2) and note the *changes of sign* of the function (cf. the examples in §§ 77–79).

76. Curve tracing. The method of curve tracing outlined in § 29 may now be somewhat strengthened, as follows:

(1) *Test the curve for symmetry with respect to the coördinate axes.* (It is symmetric with respect to Oy if the equation is unchanged when x is replaced by $-x$; etc.)

(2) *Find the points of intersection with the axes.*

(3) *Determine the behavior of y for large values of x. Find the horizontal asymptotes.*

(4) *Find the vertical asymptotes.*

(5) *Determine as closely as possible those regions of the plane in which the curve lies.*

(6) *Find the critical points; distinguish between maxima and minima.*

The above is only a general outline of the process to be followed; other steps will often suggest themselves. Free use should of course be made of the methods suggested in § 38. In many cases the points of inflection should be found and the inflectional tangents drawn, but this is not worth while if the second derivative is complicated. Translation

or rotation of axes is occasionally useful. The elementary method of tracing the curve by plotting points is too laborious to be used extensively, but it is frequently advisable to plot a few points, merely as a check on the analysis.

77. One-valued algebraic functions.

Example: Trace the curve

$$y = \frac{x}{(x+1)(x-2)}.$$

(1) There is no symmetry.

(2) The curve crosses the axes at (0, 0).

(3) As x increases through positive or negative values, y approaches 0: the line $y = 0$ is an asymptote.

(4) When $x = -1$, or $x = 2$, y is infinite: the lines $x = -1$, $x = 2$ are asymptotes.

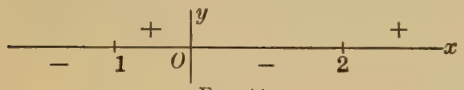


Fig. 44

(5) Since a fraction changes sign when either the numerator or the denominator does so, it appears that y changes sign as x passes through the values $0, -1, 2$. When $x > 2$, y is positive; the signs in the several intervals are therefore as shown in Fig. 44.

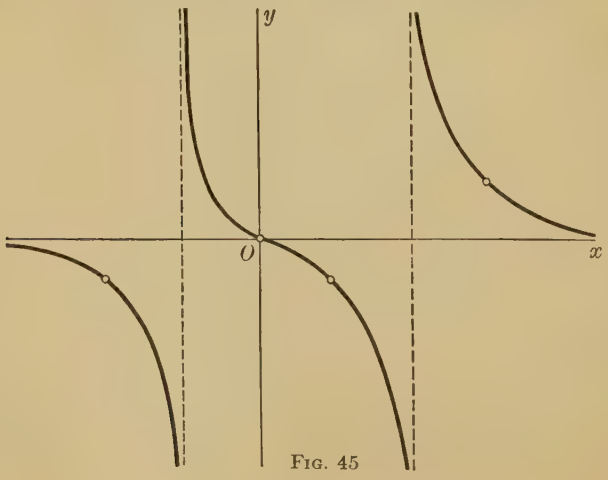


Fig. 45

(6) The derivative is

$$y' = \frac{x^2 - x - 2 - x(2x - 1)}{(x + 1)^2(x - 2)^2} = \frac{-x^2 - 2}{(x + 1)^2(x - 2)^2}$$

hence there are no critical points.

Plotting the points $(-2, -\frac{1}{2})$, $(1, -\frac{1}{2})$, $(3, \frac{3}{4})$, we draw the curve as in Fig. 45.

EXERCISES

Trace the following curves.

1. $y = \dfrac{1}{1 + x^2}$.
2. $y = \dfrac{x}{1 + x^2}$.
3. $y = \dfrac{3x}{x^2 - x + 1}$.
4. $y = \dfrac{2 - 2x}{3 + x^2}$.
5. $y = \dfrac{a^3}{a^2 - x^2}$.
6. $y = \dfrac{a^2x}{a^2 - x^2}$.
7. $y = \dfrac{2 - x}{3 - x^2}$.
8. $y = \dfrac{a^4}{(a - x)^3}$.
9. $y = \dfrac{x^3}{x^2 + a^2}$.
10. $y = \dfrac{1}{x^3 - 3x}$.
11. $y = \dfrac{a^2x}{(x - a)^2}$.
12. $y = \dfrac{x^3}{a^2 - x^2}$.
13. $y = \dfrac{2x}{x^4 + 3}$.
14. $y = \dfrac{x^2}{1 + x^4}$.
15. $y = \dfrac{4x + 4}{x^2 + 2x + 5}$.
16. $y = \dfrac{ax^2}{(x - a)^2}$.
17. $y = \dfrac{x^3}{(a - x)^2}$.
18. $y = \dfrac{4x^3}{x^4 + 27}$.
19. $y = \dfrac{ax^3}{(x - a)^3}$.
20. $y = \dfrac{1}{x^3 + 3x^2}$.
21. $y = \dfrac{2x^2}{x^2 + 2x + 3}$.
22. $y = \dfrac{(x + 1)^2}{x^2 + 2x + 2}$.
23. $y = \dfrac{3x^4 + 1}{x^4 + 1}$.
24. $y = \dfrac{(x - 2a)^3}{ax}$.
25. $y = \dfrac{1}{x(x + 1)^2}$.
26. $y = \dfrac{1}{x(x - 1)(x - 2)}$.
27. $y = \dfrac{8x - 8}{x^3}$.
28. $y = \dfrac{2x + 1}{3x^2 + 8x + 4}$.
29. $y = \dfrac{x^2 + x + 1}{x^2 + 2x + 2}$.
30. $y = \dfrac{(x^2 - 1)^2}{x}$.

31. $y = \dfrac{(x^2 - 1)^2}{x + 2}$.

32. $y = \dfrac{1}{(x^2 - 1)^2}$.

33. $y = \dfrac{2x + 3}{x^2 - 2x - 3}$.

34. $y = \dfrac{3x^4 - x^3 + 16}{3x^4 + 16}$.

35. $y = \dfrac{3x^3}{6x^2 - 9ax + 4a^2}$.

36. $y = \dfrac{-x^3}{2 + 9x + 12x^2}$.

37. $y = \dfrac{(x - 2)^3}{2x - 2}$.

38. $y = \dfrac{(x - 2a)^3}{x^2 - a^2}$.

39. $y = \dfrac{x^4}{(x - a)^3}$.

40. $y = \dfrac{(x^2 - 1)^3}{x}$.

78. Two-valued algebraic functions.

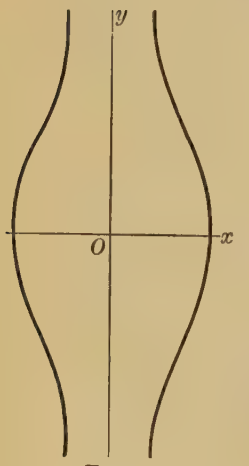


FIG. 46

Example: Trace the curve

$$y^2 = \frac{1 - x^2}{x^2}.$$

(1) The curve is symmetric with respect to both axes.

(2) The curve crosses Ox at $(\pm 1, 0)$.

(3) For large values of x, y is imaginary.

(4) The line $x = 0$ is an asymptote.

(5) The right member changes sign as x passes through 1 and through -1: y^2 is positive, and y is real, in the interval $-1 < x < 1$; y is imaginary outside that interval.

(6) Differentiating, we find

$$2yy' = \frac{-2x^3 - 2x(1 - x^2)}{x^4} = \frac{-2}{x^3},$$

or

$$y' = -\frac{1}{x^3 y};$$

hence there are no critical points. The derivative is infinite when $y = 0$, so that the curve crosses the x-axis at right angles.

The curve appears in Fig. 46.

EXERCISES

Trace the following curves.

1. $y^2 = \dfrac{1}{x+1}$.
2. $y^2 = \dfrac{1}{x^2+1}$.
3. $y^2 = \dfrac{a^3x}{a^2+x^2}$.
4. $y^2 = \dfrac{1-x}{1+x^2}$.
5. $y^2 = \dfrac{a^3x}{a^2-x^2}$.
6. $y^2 = \dfrac{a^4}{a^2-x^2}$.
7. $y^2 = x(x^2-1)$.
8. $y^2 = x(1-x^3)$.
9. $y^2 = x(x^3-1)$.
10. $y^2 = 1-x^4$.
11. $y^2 = \dfrac{1-x^2}{1+x^2}$.
12. $y^2 = \dfrac{1+x^2}{x^2}$.
13. $y^2 = \dfrac{4-x}{x}$.
14. $y^2 = \dfrac{2-x}{3-x^2}$.
15. $y^2 = \dfrac{x-2}{3-x^2}$.
16. $y^2 = \dfrac{1-x^3}{x^2}$.
17. $y^2 = \dfrac{1}{x^2(1-x)}$.
18. $y^2 = \dfrac{x^2-x}{x^2-4}$.

79. Transcendental curves.

Example: Trace the curve $y = xe^x$.

(1) There is no symmetry.

(2) The curve crosses the axes at (0, 0).

(3) As x becomes large and negative, y approaches 0 *; hence the negative x-axis is an asymptote. When x is large and positive, y is large and positive.

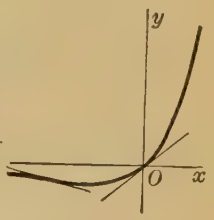


FIG. 47

(4) There are no vertical asymptotes.

(5) Since e^x is always positive, y has the same sign as x: the curve lies in the first and third quadrants.

(6) Since $y' = xe^x + e^x$, the only critical point is $\left(-1, -\dfrac{1}{e}\right)$. This is obviously a minimum point.

EXERCISES

Trace the following curves.

1. $y = e^{\frac{1}{2}x^2}$.
2. $y = e^{-\frac{1}{2}x^2}$.
3. $y = e^{\frac{1}{3}x^3}$.
4. $y = xe^{-2x}$.
5. $y = xe^{-x^2}$.
6. $y = x^2e^x$.

* Cf. § 73.

7. $y = x^3e^x$.	**8.** $y = e^x(x^2 - 3)$.	**9.** $y = x^2e^{-x^2}$.
10. $y = \frac{e^x}{x}$.	**11.** $y = e^{-x} \sin x$.	**12.** $y = 2\,e^{-x} \cos x$.
13. $y = e^{\frac{1}{x}}$.	**14.** $y = \sin^3 x$.	**15.** $y^2 = \sin x$.
16. $y = \frac{\log x}{x}$.	**17.** $y = \log^2 x$.	**18.** $y = \frac{x}{\log x}$.
19. $y^2 = \log x$.	**20.** $y = x \log x$.	**21.** $y = x \log^2 x$.
22. $y = e^{\cos x}$.	**23.** $y = x^2 \log x$.	**24.** $y = (x - 1) \log x$.
25. $y = \frac{\log x}{x^2}$.	**26.** $y = \frac{\log x}{x - 1}$.	**27.** $y = \frac{\log^2 x}{x}$.
28. $y = \log \sin x$.	**29.** $y = \log \tan x$.	**30.** $y = \log \cos 3x$.

80. Singular points. If y is defined implicitly as a function of x by the equation

$$F(x, y) = 0,$$

the derivative in general takes the form of a fraction whose numerator and denominator are functions of x and y: say

$$y' = \frac{A(x, y)}{B(x, y)}.$$

If $A(x, y)$ and $B(x, y)$ both vanish at the point $P : (x, y)$ on the curve, the slope at that point assumes the indeterminate form $\frac{0}{0}$. A point at which the derivative takes the form $\frac{0}{0}$ is called a *singular point.*

To find the singular points of a curve we must therefore find the values of x and y that satisfy the three equations

$$F(x, y) = 0, \quad A(x, y) = 0, \quad B(x, y) = 0.$$

As we have but two unknowns x and y to satisfy three equations, it appears that a curve will have singular points only under certain conditions.

It will be sufficient to consider an algebraic curve having a singular point at the origin. If a singularity occurs at any other point (h, k), the origin may be transferred to that point by the substitutions

$$x = x_1 + h, \quad y = y_1 + k.$$

81. Determination of tangents by inspection*. Let the equation of the curve be written in the form

$$F(x, y) = a_0 + b_0x + b_1y + c_0x^2 + c_1xy + c_2y^2 + \cdots + g_ny^n = 0.$$

Differentiating, we find

$$(b_0 + 2c_0x + c_1y + \cdots)dx + (b_1 + c_1x + 2c_2y + \cdots)dy = 0,$$

$$\frac{dy}{dx} = -\frac{b_0 + 2c_0x + c_1y + \cdots}{b_1 + c_1x + 2c_2y + \cdots}.$$

The origin is on the curve only if $a_0 = 0$. In that case the equation of the tangent at (0, 0) is found by the usual method (§ 22) to be

$$b_0x + b_1y = 0,$$

provided b_0 and b_1 are not both 0; *i.e.* the equation of the tangent at the origin may be found by simply equating to 0 the group of terms of the first degree.

In case a_0, b_0 and b_1 are all 0, the origin is on the curve and the derivative is indeterminate at that point; hence the origin is a singular point. In this case the equation of the curve evidently contains no terms of lower degree than the second.

For convenience let us put †

$$c_0x^2 + c_1xy + c_2y^2 = c_2\left(y^2 + \frac{c_1}{c_2}xy + \frac{c_0}{c_2}x^2\right)$$

$$= c_2(y - m_1x)(y - m_2x).$$

Then

$$F(x, y) = c_2(y - m_1x)(y - m_2x) + d_0x^3 + \cdots = 0.$$

The abscissas of the points of intersection of the line

$$y = mx$$

with this curve are given by the equation

$$c_2x^2(m - m_1)(m - m_2) + x^3(d_0 + \cdots) + \cdots = 0.$$

Two roots of this equation are 0: *every* line $y = mx$ intersects the curve in two coincident points at the origin. But

* The argument of this article applies only to *algebraic* curves.

† The argument is readily modified to take care of the case $c_2 = 0$.

the above equation in x also shows that if we let m approach either m_1 or m_2, the coefficient of x^2 approaches 0; *i.e.* a *third* point of intersection of the curve with the line $y = mx$ approaches the origin, and the lines

$$y = m_1x, \quad y = m_2x$$

are both tangent to the curve at the singular point. These lines may of course be real and distinct, real and coincident, or imaginary.

Since

$$c_2(y - m_1x)(y - m_2x) = c_0x^2 + c_1xy + c_2y^2,$$

we see that the equations of the two tangents are obtained by *equating the second degree terms to* 0, and factoring the left member of the resulting equation.

The argument we have used can be extended to show that if $F(x, y)$ has no terms of degree lower than the k-th, any line through the origin meets the curve there in k points, and the k tangents to the curve at the origin are obtained by equating the group of terms of lowest degree to 0.

A point at which there are two tangents (whether distinct, coincident, or imaginary) is called a *double point;* one at which there are three tangents is a *triple point;* etc. For the sake of simplicity we shall confine our attention, except where the contrary is indicated, to curves whose only singularity is a double point at the origin.

In summary, we have the following

THEOREM: *If the equation* $F(x, y) = 0$ *contains terms of the second, but none of lower degree, the origin is a double point; the tangents at that point are found by equating to* 0, *and factoring, the group of terms of second degree.*

As noted above, every line through a double point has at that point two coincident intersections with the curve. Hence a conic can have no double point, since a line through that point and another point of the conic would have three intersections with the curve. Similarly, a cubic can have no triple point, and not more than one double point, etc.

82. Classification of double points. If the tangents at a double point are real and different, the point is called a *node:* two branches of the curve cross each other, as in Fig. 48.

FIG. 48

If the tangents are imaginary, the point is called an *isolated point,* or *conjugate point:* there is no other portion of the curve in its vicinity. Such a point is P in Fig. 49.

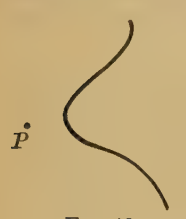


FIG. 49

FIG. 50

FIG. 51

If the tangents are real and coincident, the point is in general * either a *cusp* † (Fig. 50) or a *double cusp* (Fig. 51).

Example: Trace the curve

$$y^2 = x^4 - x^6.$$

(1) The curve is symmetric with respect to both axes.

(2) The curve crosses the axes at $(0, 0)$, $(\pm 1, 0)$.

(3) When x is large, y is imaginary.

(4) There are no vertical asymptotes.

(5) Writing the equation in the form

$$y^2 = x^4 (1 - x^2),$$

we see that y is real in the interval $-1 < x < 1$, and imaginary outside that region.

(6) Differentiating, we get

$$2yy' = 4x^3 - 6x^5,$$

or

$$y' = \frac{x^3(2 - 3x^2)}{y}.$$

* In rare instances it may be an isolated point.

† If the branches lie on opposite sides of the cuspidal tangent, as in Fig. 50, the point is a *cusp of the first kind;* if on the same side, a *cusp of the second kind.* The latter singularity is comparatively uncommon.

Equating this to 0, we find the critical values $x = \pm\sqrt{\frac{2}{3}}$. It should be noted that when $x = 0$, we have also $y = 0$, and y' takes the form $\frac{0}{0}$.

(7) As a seventh step we note, either from the result just obtained or by the theorem of § 81, that the origin is a singular point. The tangents at the origin are given by the equation

$$y^2 = 0;$$

that is, they coincide in the x-axis. Thus the point is either a cusp or a double cusp; step (1) shows that it must be the latter.

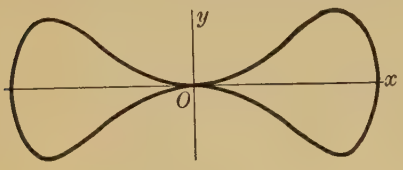


FIG. 52

The curve is drawn in Fig. 52.

EXERCISES

Trace the following curves.

1. $y^2 = x^2(1 - x)$.
2. $ay^2 = x^3$.
3. $y^2 = x^2(x - 1)$.
4. $a^2y^2 = x^2(x^2 - a^2)$.
5. $y^2(x^2 + a^2) = a^2x^2$.
6. $a^2y^2 = x^3(x - a)$.
7. $y^2 = x^4(1 - x)$.
8. $y^2(x^2 + y^2) = a^2x^2$.
9. $x^2y^2 = a^2(x^2 + y^2)$.
10. $a^2y^2 = x^2(a^2 - x^2)$.
11. $y^2 = \dfrac{x^3}{2a - x}$.
12. $y^2 = \dfrac{x^2}{1 - x}$.
13. $y^3 = \dfrac{x^2}{(1 - x)^2}$.
14. $y^2 = \dfrac{x^2}{1 - x^2}$.
15. $y^3 = 3ax^2 - x^3$.
16. $ay^3 = x^2(a^2 - x^2)$.
17. $y^2 = \dfrac{x^2(x^2 - a^2)}{x^2 + a^2}$.
18. $y^2 = \dfrac{ax^3}{x^2 - a^2}$.
19. $y^2 = \dfrac{x^2(1 + x)}{1 - x}$.
20. $y^3 = \dfrac{x^2}{1 - x^2}$.

Show that the following curves have singular points as indicated, and trace the curves.

21. $y^3 = x^3(1 + x)$: triple point at (0, 0).
22. $y^3 = \dfrac{x^4}{a + x}$: triple point at (0, 0).

23. $x^2y^2 = (x + a)^2(4a^2 - x^2)$: node at $(-a, 0)$.

24. $ay^3 = (x^2 - a^2)^2$: cusps at $(\pm a, 0)$.

25. $y^2 = x(x^2 - 1)^2$: node at $(1, 0)$, isolated point at $(-1, 0)$.

26. Prove the statements in the last sentence of § 81.

27. Prove that a curve of the fourth degree cannot have more than three double points. (Assume four; then consider the conic through these four points and a fifth point of the curve.)

28. Prove that the graph of a one-valued algebraic function

$$y = f(x)$$

cannot have any singular points.

83. Curve tracing in polar coördinates. For the sake of brevity, we will omit a discussion of the use of the calculus in tracing polar curves. By way of review, however, we give below a list of curves to be traced by the methods of analytic geometry.

EXERCISES

Trace the following curves.

1. $r = 2a\cos\theta$.

2. $r = a\sin^2\theta$.

3. $r = a\tan\theta$.

4. $r = a\sec\theta\tan\theta$.

5. $r = a(1 + \cos\theta)$.

6. $r = a(2 + \cos\theta)$.

7. $r = \dfrac{1}{1 - \sin\theta}$.

8. $r = \dfrac{1}{2 - \cos\theta}$.

9. $r = a\sin 2\theta$.

10. $r = a\cos 2\theta$.

11. $r = a\sin 3\theta$.

12. $r = a\cos 4\theta$.

13. $r^2 = a^2\sin\theta$.

14. $r^2 = a^2\cos 2\theta$.

15. $r^2\sin 2\theta = a^2$.

16. $r = a(1 + \sec\theta)$.

CHAPTER XII

THE INDEFINITE INTEGRAL

84. Integration. We have been occupied up to this point with the problem: Given a function, to find its derivative. Many of the most important applications of the calculus lead to the inverse problem: Given the derivative of a function, to find the function. The required function is called an *integral* of the given derivative, or *integrand*, and the process of finding it is called *integration*.

If $f(x)$ is a given function and $F(x)$ is a function whose derivative is $f(x)$, the relation between them is expressed by writing

$$F(x) = \int f(x)\,dx,$$

where the "integral sign" $\int$ indicates that we are to perform the operation of integration upon $f(x)\,dx$. For reasons that will appear later, it is customary to write after the integral sign the differential $f(x)\,dx$ rather than the derivative $f(x)$.

Examples: (*a*) Find the equation of a curve whose slope at every point is equal to twice the abscissa of the point.

We have to find a function y such that

$$\frac{dy}{dx} = 2x, \text{ or } dy = 2x\,dx;$$

hence

$$y = \int 2x\,dx.$$

It appears at once that $2x$ is the derivative of x^2. Thus a curve having the desired property is the parabola

$$y = x^2.$$

But it is clear that if

$$y = x^2 + C, \tag{1}$$

where C is any constant whatever, we still have

$$dy = 2x\,dx,$$

and the given condition is satisfied by any one of the *family* of parabolas represented by (1). In order to obtain a unique answer to our problem, we must have some additional information about the curve. Thus, if it is to pass through the point P : (1, 2), we substitute these coördinates in (1):

$$2 = 1 + C,\ C = 1,$$

and the answer is

$$y = x^2 + 1.$$

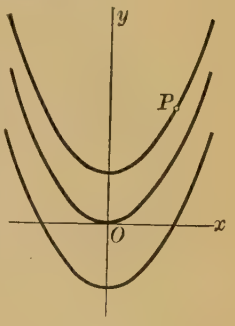


FIG. 53

(*b*) Find the velocity of a body falling freely under gravity at the end of 5 seconds, if the initial velocity is 20 ft. per sec. upward.

Taking motion downward as positive, we have to find v from the relation (see § 66)

$$j = \frac{dv}{dt} = g.$$

Hence

$$v = \int g\,dt = gt + C.$$

Making use of the fact that $v = -20$ when $t = 0$, we get

$$-20 = 0 + C,$$

and the velocity at the time t is

$$v = gt - 20.$$

At the end of 5 seconds we have, taking $g = 32$,

$$v = 140 \text{ ft. per sec.}$$

85. Integration an indirect process. Differentiation is a direct process; by means of the fundamental formulas the derivative of any elementary function may be found. On the other hand, to find an integral of a given function, we must be able to discover a function whose derivative is the given integrand, and this is always in the last analysis a matter of trial. The problem can by no means always be solved; in fact, there are many comparatively simple functions whose integrals cannot be expressed in terms of elementary functions *.

86. Constant of integration. In each of the examples of § 84, an arbitrary constant presented itself. It is clear that this will be the case in general; *i.e.* a function whose derivative is given is not completely determined, since it contains an arbitrary additive constant, the *constant of integration.*

On account of the presence of this undetermined constant, the function $\int f(x)\,dx$ is called the *indefinite integral* of $f(x)$.

87. Functions having the same derivative. In § 86 it was tacitly assumed that if the derivative of a function is given, the function is determined aside from an additive constant. That this is true follows from the

THEOREM: *Two functions having the same derivative differ only by a constant.*

The theorem is almost self-evident. Let $\phi(x)$ and $\psi(x)$ be the two functions, and place

$$y = \phi(x) - \psi(x).$$

By hypothesis,

$$\frac{dy}{dx} = \phi'(x) - \psi'(x) = 0.$$

The rate of change of y with respect to x is everywhere 0, hence y is constant.

* It will be shown in § 110 that for every continuous function *an integral exists*, although it may not be an elementary function.

EXERCISES

Evaluate the following integrals, checking the answer in each case by differentiation.

1. $\int x^3\,dx$.
2. $\int (x-3)\,dx$.
3. $\int (1-2x)\,dx$.
4. $\int (t^2-3)\,dt$.
5. $\int \frac{dx}{x^2}$.
6. $\int (1-y)^2\,dy$.
7. $\int \frac{dx}{x}$.
8. $\int \frac{dx}{1+x}$.
9. $\int \frac{dz}{1-z}$.
10. $\int \sqrt{x+1}\,dx$.
11. $\int \sqrt{1-x}\,dx$.
12. $\int \sin\theta\,d\theta$.
13. $\int \sin 2\theta\,d\theta$.
14. $\int e^{2u}du$.
15. $\int \sec^2\phi\,d\phi$.
16. $\int (1+x)^6\,dx$.
17. $\int (1+3x)^6\,dx$.
18. $\int \frac{dt}{1+t^2}$.
19. $\int \frac{du}{\sqrt{1-u^2}}$.
20. $\int \cos\frac{\theta}{2}\,d\theta$.
21. $\int \sin\theta\cos\theta\,d\theta$.

22. Find the equation of the family of curves whose slope at every point is equal to the square of the abscissa of the point. Exhibit graphically.

23. Find the equation of that one of the curves of Ex. 22 that passes through the point $(3, -5)$. *Ans.* $y = \frac{1}{3}x^3 - 14$.

24. Find the equation of the curve for which $y'' = 4$, if the curve touches the line $y = 3x$ at $(2, 6)$. *Ans.* $y = 2x^2 - 5x + 8$.

25. Find the curve for which $y'' = 2$, and which passes through $(3, 1)$, $(-1, 2)$. *Ans.* $4y = 4x^2 - 9x - 5$.

26. Find the curve for which $y'' = 12x$, and which touches the line $y = 3x - 1$ at $(1, 2)$.

27. Find the curve for which $y'' = 6x$, and which passes through $(1, 2)$ and $(2, 3)$. *Ans.* $y = x^3 - 6x + 7$.

28. Find the curve for which $y'' = -\frac{1}{x^2}$, and which touches the line $y = x - 1$ at $(1, 0)$.

29. Find the curve for which $y'' = \frac{1}{x^3}$, and which passes through $(1, 1)$, $(-1, 2)$. *Ans.* $y = \frac{1}{2x} - x + \frac{3}{2}$.

30. Find the curve for which $y'' = e^x$, and which touches the x-axis at $(0, 0)$. *Ans.* $y = e^x - x - 1$.

CHAPTER XIII

STANDARD FORMULAS OF INTEGRATION

88. Standard formulas. If we were to try at present to solve any but the simplest applied problems involving integration, we must fail through inability to evaluate the indefinite integrals involved. We shall therefore devote this and the next three chapters to the technique of integration — the formal evaluation of indefinite integrals — after which the question of applications will be taken up.

As the first step toward facility in integration, the student must become thoroughly familiar with the following

FUNDAMENTAL INTEGRATION FORMULAS

$$(1)\quad \int du = u + C,$$

$$(2)\quad \int (du + dv) = \int du + \int dv,$$

$$(3)\quad \int c\,du = c\int du,$$

$$(4)\quad \int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad (n \neq -1)$$

$$(5)\quad \int \frac{du}{u} = \log u + C,$$

$$(6)\quad \int e^u\,du = e^u + C,$$

$$(7)\quad \int \cos u\,du = \sin u + C,$$

$$(8)\quad \int \sin u\,du = -\cos u + C,$$

$$(9)\quad \int \sec^2 u\,du = \tan u + C,$$

$$(10)\quad \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C,$$

$$(11)\quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C,$$

$$(12)\quad \int u\, dv = uv - \int v\, du.$$

It is strongly recommended that each of the formulas be written out by the student in words, and memorized in that form.

The test of the correctness of an integral is that its derivative must be the given integrand. The above formulas may be verified at once by differentiation.

89. Formulas (1)–(3). Formula (**1**) merely embodies the definition of an integral.

Formula (**2**) is readily extended to the case of any number of terms. The formula shows that if the integrand consists of a sum of terms, each term may be integrated separately.

Formula (**3**) says that if the integrand contains a constant factor, that factor may be written before the integral sign. As a corollary, we may *introduce a constant factor into the integrand*, provided we place its reciprocal before the integral sign. But the student must beware of introducing *variable* factors by this rule.

90. Formula (4): Powers. This formula evidently fails when $n = -1$. The exceptional case $\int \frac{du}{u}$ is taken care of by formula (**5**).

Examples:

$$(a)\quad \int \left(3x^3 + 1 + \frac{1}{2x^2}\right) dx = 3\int x^3\, dx + \int dx + \frac{1}{2}\int x^{-2}\, dx$$

$$= \frac{3x^4}{4} + x - \frac{1}{2}x^{-1} + C$$

$$= \frac{3x^4}{4} + x - \frac{1}{2x} + C.$$

$$(b)\quad \int x(1+x^2)^2\,dx = \int (x + 2x^3 + x^5)\,dx$$
$$= \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{6} + C.$$

A better method: Introducing a factor 2 after the integral sign and its reciprocal in front, we have

$$\int x(1+x^2)^2\,dx = \frac{1}{2}\int (1+x^2)^2 \cdot 2x\,dx.$$

Since $2x\,dx$ is the differential of $1 + x^2$, formula (**4**) applies with $u = 1 + x^2$, $n = 2$. Hence

$$\frac{1}{2}\int (1+x^2)^2 \cdot 2x\,dx = \frac{1}{2}\frac{(1+x^2)^3}{3} + C = \frac{1}{6}(1+x^2)^3 + C.$$

The student should compare the two answers that have been obtained in this example.

$$(c)\quad \int \sqrt{a+bx}\,dx = \frac{1}{b}\int \sqrt{a+bx}\cdot b\,dx$$
$$= \frac{1}{b}\frac{(a+bx)^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$

Here the quantity $a + bx$ is taken as the variable of integration, the factor b is introduced to give the proper differential, and (**4**) then applies with $n = \frac{1}{2}$.

EXERCISES

Evaluate the following integrals; check the answers by differentiation.

1. $\int (3x^3 - 2x^2 + 4x - 1)\,dx.$
2. $\int (x^4 - 2x^2 + 3)\,dx.$
3. $\int \left(2x - \frac{1}{x^2}\right) dx.$
4. $\int (\sqrt{x} - 2\sqrt{x^3})\,dx.$
5. $\int \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right) dy.$
6. $\int \left(\frac{2}{x^3} - \frac{3}{x^2}\right) dx.$
7. $\int (2-x)^5\,dx.$
8. $\int 3(y+1)^3\,dy.$
9. $\int \sqrt{1+t}\,dt.$
10. $\int \sqrt{1-3t}\,dt.$

11. $\int x(a^2 - x^2)^2\,dx.$

12. $\int x^2(1 - x^2)\,dx.$

13. $\int (a^2 - x^2)^2\,dx.$

14. $\int \frac{dz}{(1 + z)^3}.$

15. $\int \frac{dz}{(1 - z)^3}.$

16. $\int u^3(2u - 3)\,du.$

17. $\int u(u - 2)^2\,du.$

18. $\int \frac{dx}{\sqrt{1 - 2x}}.$

19. $\int (a - x)^{\frac{3}{2}}\,dx.$

20. $\int \frac{v^2\,dv}{\sqrt{1 + v^3}}.$

21. $\int \frac{r\,dr}{(a^2 - r^2)^2}.$

22. $\int x\sqrt{(a^2 - x^2)^3}\,dx.$

23. $\int (x + 1)\sqrt{x}\,dx.$

24. $\int (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2\,dx.$

25. $\int (x + 2)(x^2 + 3)\,dx.$

26. $\int \sin\theta \cos\theta\,d\theta.$

27. $\int \cos^2\theta \sin\theta\,d\theta.$

28. $\int \sin 3\theta \cos 3\theta\,d\theta.$

29. $\int e^y\sqrt{1 + e^y}\,dy.$

30. $\int \frac{\sin\phi\,d\phi}{\cos^2\phi}.$

31. $\int (1 - z^2)^3\,dz.$

32. $\int \frac{v\,dv}{\sqrt{a^2 + v^2}}.$

33. $\int \frac{x^2 - 3x + 2}{\sqrt{x}}\,dx.$

34. $\int \frac{dy}{\sqrt{(3y - 2)^3}}.$

35. $\int \cos\theta\sqrt{1 + \sin\theta}\,d\theta.$

36. $\int e^x(1 - 2e^x)^2\,dx.$

37. Find the equation of the family of curves for which $y'' = (x + 1)^2$ and determine that one of the family that touches the line $x + 3y = 0$ at (0, 0). *Ans.* $y = \frac{1}{12}(x + 1)^4 - \frac{2}{3}x - \frac{1}{12}.$

38. Find the equation of the family of curves for which $y'' = \frac{2}{(1 - x)^3}$, and determine that one of the family that passes through (0, 0), (2, 4).

39. A body moves under an acceleration $\frac{d^2x}{dt^2} = 2t - 7.$ If $v = 6$ and $x = 0$ when $t = 0$, find v and x in terms of t, and describe the motion. *Ans.* $x = \frac{1}{3}t^3 - \frac{7}{2}t^2 + 6t.$

40. A body moves under an acceleration $\frac{dv}{dt} = \frac{8}{(t + 1)^3}.$ If $v = -3$ and $x = 0$ when $t = 0$, find v and x in terms of t, and describe the motion.

91. Formulas (5)–(6): Logarithms and exponentials.

Examples: (*a*) Evaluate $\int \frac{x\,dx}{a^2 + x^2}$.

If the factor 2 be inserted after the integral sign, the numerator becomes the differential of the denominator, and (**5**) applies:

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\int \frac{2\,x\,dx}{a^2 + x^2} = \frac{1}{2}\log\,(a^2 + x^2) + C.$$

(*b*) Evaluate $\int \frac{x^2 - x}{x + 1}\,dx$.

By division we find

$$\frac{x^2 - x}{x + 1} = x - 2 + \frac{2}{x + 1}.$$

Whence

$$\int \frac{x^2 - x}{x + 1}\,dx = \int \left(x - 2 + \frac{2}{x + 1}\right) dx$$
$$= \tfrac{1}{2}\,x^2 - 2\,x + 2\log\,(x + 1) + C.$$

(*c*) Evaluate $\int e^{3x}\,dx$.

If the factor 3 be inserted, this is integrable by (**6**):

$$\int e^{3x}\,dx = \tfrac{1}{3}\int e^{3x} \cdot 3\,dx = \tfrac{1}{3}\,e^{3x} + C.$$

In (*b*) we have used a device that finds frequent application:

As the first step toward integrating a rational fraction, carry out the indicated division until the numerator is of lower degree than the denominator.

EXERCISES

Evaluate the following integrals; check the results.

1. $\int \frac{dx}{x - 3}$.
2. $\int \frac{dx}{3\,x - 5}$.
3. $\int \frac{2\,du}{5 - 4\,u}$.
4. $\int \frac{dy}{3(1 - y)}$.
5. $\int \frac{dx}{\sqrt{2\,x + 1}}$.
6. $\int \frac{dx}{\sqrt{5 - 4\,x}}$.

7. $\int \frac{t\,dt}{a^2 - t^2}$.

8. $\int \frac{y^2\,dy}{1 - 2y^3}$.

9. $\int \frac{dz}{(1 - z)^2}$.

10. $\int \frac{3x - 1}{2x + 2}\,dx$.

11. $\int \frac{x\,dx}{\sqrt{a^2 + x^2}}$.

12. $\int \frac{y\,dy}{(2 - 3y^2)^2}$

13. $\int \frac{dx}{x^2 - 4x + 4}$.

14. $\int \tan\theta\,d\theta$.

15. $\int \frac{\sin\theta\,d\theta}{1 + \cos\theta}$.

16. $\int \frac{\sin x\,dx}{\cos^3 x}$.

17. $\int e^{\frac{x}{2}}\,dx$.

18. $\int e^{-5t}\,dt$.

19. $\int xe^{-x^2}\,dx$.

20. $\int \frac{e^x\,dx}{e^x - 1}$.

21. $\int \frac{(e^{\frac{x}{3}} - e^{-\frac{x}{3}})^2}{3}\,dx$.

22. $\int \frac{3x - 5}{1 - 2x}\,dx$.

23. $\int \frac{x^2 - x + 2}{x + 3}\,dx$.

24. $\int \frac{x^3\,dx}{x^2 + 1}$.

25. $\int \frac{e^{2x}\,dx}{(1 - e^{2x})^2}$.

26. $\int \frac{\cos 2\phi\,d\phi}{2\sin 2\phi - 3}$.

27. $\int \frac{e^x + 1}{e^x}\,dx$.

28. $\int \frac{(4x + 2)\,dx}{x^2 + x - 1}$.

29. $\int \frac{y^2\,dy}{\sqrt{3 - 2y^3}}$.

30. $\int t^3 e^{-2t^4}\,dt$.

31. $\int \sqrt[3]{\frac{a}{x}}\,dx$.

32. $\int \frac{a\,dy}{(a - y)^2}$.

33. A body moves with an acceleration $\frac{d^2x}{dt^2} = \frac{4}{(1 + 2t)^2}$. Given $v = x = 0$ when $t = 0$, find v and x in terms of t, and describe the motion.

Ans. $x = 2t - \log(1 + 2t)$.

34. A body moves with an acceleration $\frac{d^2x}{dt^2} = -\frac{2}{(1 + t)^2}$. Given $v = 1$ and $x = 0$ when $t = 0$, discuss the motion.

35. A body moves with an acceleration $j = 2e^{-t}$. Given $x = 0$, $v = 30$ when $t = 0$, discuss the motion.

Ans. $x = 32t - 2 + 2e^{-t}$.

36. A body moves with an acceleration $j = -8e^{-\frac{t}{2}}$. If $x = 0$, $v = 8$ when $t = 0$, discuss the motion.

Ans. $x = 32(1 - e^{-\frac{t}{2}}) - 8t$.

92. Formulas (7)–(9): Trigonometric functions.

EXERCISES

Evaluate the following integrals.

1. $\int \sin 2x\, dx.$
2. $\int \sec^2 \frac{\theta}{2}\, d\theta.$
3. $\int x \sin x^2\, dx.$
4. $\int \cos kt\, dt.$
5. $\int \frac{\sin 3\theta\, d\theta}{1 - 2\cos 3\theta}.$
6. $\int \frac{\cos x\, dx}{\sin^3 x}.$
7. $\int \tan^2 \theta\, d\theta.$
8. $\int \tan 5\theta\, d\theta.$
9. $\int e^x \cos e^x\, dx.$
10. $\int \sin^2 2\alpha \cos 2\alpha\, d\alpha.$
11. $\int (1 + \cos 2\theta)\, d\theta.$
12. $\int \sin\theta(1 - 2\cos\theta)^2\, d\theta.$
13. $\int \frac{d\theta}{\sin^2 \theta}.$
14. $\int e^{\cos t} \sin t\, dt.$
15. $\int \frac{\sin x\, dx}{\cos^5 x}.$
16. $\int \frac{2\sin\theta\, d\theta}{3\cos\theta - 1}.$
17. $\int \frac{d\theta}{\sin\theta \tan\theta}.$
18. $\int \csc^2\theta \cot\theta\, d\theta.$
19. $\int \frac{1 + \cos x}{\sin^2 x}\, dx.$
20. $\int \frac{\cos x - \sin x}{\sin x}\, dx.$

21. The velocity of a moving body is

$$v = 2\cos 2t.$$

Find the acceleration, and the distance from the starting point, in terms of t. Describe the motion.

22. The acceleration of a moving body is

$$j = -4\cos 2t.$$

If $v = 0$ and $x = 1$ when $t = 0$, describe the motion.

23. The acceleration of a moving body is

$$j = -3\sin 3t.$$

If $v = 1$, $x = 0$ when $t = 0$, study the motion.

24. The acceleration of a moving body is

$$j = \sin \tfrac{1}{2}t.$$

If $v = x = 0$ when $t = 0$, study the motion.

Ans. $x = 2t - 4\sin \frac{1}{2}t.$

93. Formulas (10)–(11): Inverse trigonometric functions.

Example:

$$\int \frac{dx}{x^2 + 2x + 5} = \int \frac{dx}{(x+1)^2 + 4} = \tfrac{1}{2} \arctan \frac{x+1}{2} + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{\sqrt{9 - x^2}}$.
2. $\int \frac{x\,dx}{\sqrt{1 - x^4}}$.
3. $\int \frac{dx}{4x^2 + 9}$.
4. $\int \frac{2\,dx}{9x^2 + 5}$.
5. $\int \frac{dx}{\sqrt{1 - 2x^2}}$.
6. $\int \frac{du}{\sqrt{3 - 4u^2}}$.
7. $\int \frac{dx}{x^2 + 6x + 13}$.
8. $\int \frac{dt}{2t^2 + 2t + 1}$.
9. $\int \frac{dz}{3z^2 + 6z + 4}$.
10. $\int \frac{du}{\sqrt{3 + 2u - u^2}}$.
11. $\int \frac{(x+1)\,dx}{x^2 + 4}$.
12. $\int \frac{x\,dx}{\sqrt{5 - 2x^2}}$.
13. $\int \frac{e^{2u}\,du}{\sqrt{1 + e^{2u}}}$.
14. $\int \frac{(x+2)\,dx}{x^2 + 4x + 3}$.
15. $\int \frac{\sin x\,dx}{1 + \cos^2 x}$.
16. $\int \frac{3x + 2}{x^2 + 2}\,dx$.
17. $\int \frac{x^2 - 3x - 5}{x^2 + 4}\,dx$.
18. $\int \frac{x^3\,dx}{1 + 2x^2}$.
19. $\int \frac{x\,dx}{(1 - 2x^2)^2}$.
20. $\int \frac{du}{\sqrt{-3 - 12u - 9u^2}}$.

94. Formula (12): Integration by parts. From the formula for the differential of a product,

$$d(uv) = u\,dv + v\,du,$$

we find, integrating both sides,

$$uv = \int u\,dv + \int v\,du.$$

Transposing, we obtain formula (12):

$$\int u\,dv = uv - \int v\,du.$$

Integration by this formula is called *integration by parts.* The use of the formula will be explained by the following

Examples: (*a*) Evaluate $\int xe^x\,dx$.

Let

$$u = x, \quad dv = e^x\,dx,$$

whence

$$du = dx, \quad v = \int e^x\,dx = e^x.$$

Then $$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

(*b*) Evaluate $\int \log x\,dx$.

Let

$$u = \log x, \quad dv = dx,$$

whence

$$du = \frac{dx}{x}, \qquad v = x.$$

Then

$$\int \log x\,dx = x \log x - \int x \cdot \frac{dx}{x} = x \log x - x + C.$$

(*c*) Evaluate $\int e^x \sin x\,dx$.

Let

$$u = e^x, \qquad dv = \sin x\,dx,$$

so that

$$du = e^x\,dx, \quad v = -\cos x.$$

Then

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx. \tag{1}$$

The new integral thus obtained is no easier to evaluate than the original one. Let us, however, return to the original integral and take

$$u = \sin x, \quad dv = e^x\,dx,$$

whence

$$du = \cos x\,dx, \quad v = e^x,$$

and

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx. \tag{2}$$

Eliminating the quantity $\int e^x \cos x\, dx$ by adding (1) and (2), we get

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x,$$

whence

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x).$$

Integration by parts is highly important, as it succeeds in many cases when the methods of the preceding articles fail. The success of the method depends as a rule on our ability to choose u and dv so that the integral $\int v\, du$ is easier to evaluate than the given one. No general directions can be given for choosing u and dv; if the new integral is no simpler than the original, we should begin over again with a different choice of u and dv.

EXERCISES

Evaluate the following integrals.

1. $\int xe^{2x}\, dx.$
2. $\int t \sin t\, dt.$
3. $\int x^2 e^{2x}\, dx.$ *Ans.* $\tfrac{1}{2} x^2 e^{2x} - \tfrac{1}{2} xe^{2x} + \tfrac{1}{4} e^{2x} + C.$
4. $\int z^2 \log z\, dz.$
5. $\int x\sqrt{x^2 + 3}\, dx.$
6. $\int u\sqrt{1 + u}\, du.$
7. $\int u\sqrt{1 - 2u}\, du.$
8. $\int xe^{-2x^2}\, dx.$
9. $\int \frac{x\, dx}{\sqrt{1 - x}}.$
10. $\int \theta(1 - \cos\theta)\, d\theta.$
11. $\int \frac{y\, dy}{(1 - y)^2}.$
12. $\int \arcsin x\, dx.$
13. $\int x^3\sqrt{a^4 - x^4}\, dx.$
14. $\int x^3\sqrt{a^2 - x^2}\, dx.$ *Ans.* $-\tfrac{1}{3} x^2(a^2 - x^2)^{\frac{3}{2}} - \tfrac{2}{15}(a^2 - x^2)^{\frac{5}{2}} + C.$

15. $\int \frac{y^3\,dy}{\sqrt{a^2 - y^2}}$. *Ans.* $-y^2(a^2 - y^2)^{\frac{1}{2}} - \frac{2}{3}(a^2 - y^2)^{\frac{3}{2}} + C$.

16. $\int x^2 \arcsin x\,dx$.

Ans. $\frac{1}{3}x^3 \arcsin x + \frac{1}{3}x^2(1 - x^2)^{\frac{1}{2}} + \frac{2}{9}(1 - x^2)^{\frac{3}{2}} + C$.

17. $\int x \sec^2 x\,dx$.

18. $\int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{3}{2}}}$.

19. $\int \frac{x^2\,dx}{(1 + x^2)^2}$.

20. $\int \sin^3 \theta\,d\theta$.

21. $\int \sec^4 \theta\,d\theta$.

22. $\int e^{\theta} \cos \theta\,d\theta$.

23. $\int e^{-\theta} \sin 2\theta\,d\theta$.

24. $\int e^{3\theta} \cos 2\theta\,d\theta$.

25. $\int x \sinh x\,dx$.

26. $\int x \cosh 2x\,dx$.

MISCELLANEOUS EXERCISES

Evaluate the following integrals.

1. $\int \cot \theta\,d\theta$.

2. $\int \frac{x\,dx}{\sqrt{5 - 6x^2}}$.

3. $\int \frac{x^3 - x}{2 + 3x}\,dx$.

4. $\int x \cos 3x\,dx$.

5. $\int \frac{e^x\,dx}{e^{2x} + 1}$.

6. $\int \frac{\log x}{x}\,dx$.

7. $\int \left(1 + \cos \frac{\theta}{2}\right)^3 \sin \frac{\theta}{2}\,d\theta$.

8. $\int x(1 - x^4)\,dx$.

9. $\int xe^{-x^2}\,dx$.

10. $\int \sin^2 2x \cos 2x\,dx$.

11. $\int \frac{\sec^2 2\theta\,d\theta}{1 - \tan 2\theta}$.

12. $\int \frac{e^x\,dx}{1 - 3e^x}$.

13. $\int \left(h - \frac{hy}{a}\right)^2 dy$.

14. $\int x(4 - x^2)^4\,dx$.

15. $\int \frac{dx}{(1 - x)^4}$.

16. $\int \frac{x\,dx}{(1 - x)^4}$.

17. $\int \frac{dx}{x^2 + 8x + 20}$.

18. $\int \cos \theta \log \sin \theta\,d\theta$.

19. $\int \frac{dx}{x \log^2 x}$.

20. $\int \frac{x^2\,dx}{\sqrt{1 - x^6}}$.

21. $\int \log (1 - 2x)\,dx$.

22. $\int \frac{(x - 2)\,dx}{x^2 + 3}$.

23. $\int e^{\tan t} \sec^2 t \, dt.$

24. $\int \frac{x^3 + x^2 + x}{x^2 + 9} \, dx.$

25. $\int (a^{\frac{2}{3}} - y^{\frac{2}{3}})^3 \, dy.$

26. $\int \frac{(x + x^3) \, dx}{\sqrt{4 - x^4}}.$

27. $\int \frac{x^3 \, dx}{(1 - x^2)^2}.$

28. $\int x \tan^2 x \, dx.$

29. $\int \arctan x \, dx.$

30. $\int \frac{\arctan x}{1 + x^2} \, dx.$

31. $\int \frac{dx}{\sqrt{2x - x^2}}.$

32. $\int y\left(b - \frac{by}{a}\right)^3 dy.$

33. $\int \frac{x - x^2}{1 + x^2} \, dx.$

34. $\int e^{-x}(1 + e^{-x})^3 \, dx.$

35. $\int \frac{x(1 + x^2)}{1 + x} \, dx.$

36. $\int \sec^2 \theta \tan^3 \theta \, d\theta.$

37. $\int (1 - x^3)^2 \, dx.$

38. $\int (a - x)^{\frac{5}{2}} \, dx.$

39. $\int \frac{dx}{x \log x}.$

40. $\int \frac{dx}{x^2 - 6x + 10}.$

41. $\int x(a - x)^{\frac{5}{2}} \, dx.$

42. $\int x^2 \sin x \, dx.$

43. $\int \frac{x^3 \, dx}{1 + x^8}.$

44. $\int \frac{1 - e^{-x}}{1 + e^{-x}} \, dx.$

45. $\int \frac{dx}{(3x - 4)^2}.$

46. $\int \frac{\sqrt{1 + \log x}}{x} \, dx.$

47. $\int \frac{x^2 \, dx}{1 - 3x^3}.$

48. $\int \frac{2x^2 - 3x - 1}{x^2 + 3} \, dx.$

49. $\int \theta \csc^2 \theta \, d\theta.$

50. $\int \cos^3 \theta \, d\theta.$

CHAPTER XIV

TRIGONOMETRIC INTEGRALS

95. Introduction. A great many integrals cannot be reduced directly to a standard form, but can be evaluated after a suitable transformation of the integrand. Some of the more important of these integrals will be considered in this and the next two chapters.

96. Transformation by the identities of trigonometry.

Type I: $\int \sin^m x \cos^n x\, dx$, *where either* m *or* n *is a positive odd integer.*

For definiteness, let n be a positive odd integer. Writing the integral in the form $\int \sin^m x \cos^{n-1} x \cdot \cos x\, dx$, and putting

$$\cos^2 x = 1 - \sin^2 x,$$

we obtain a series of powers of $\sin x$ each multiplied by $\cos x\, dx$, and therefore integrable at once. Similarly when m is positive and odd.

Example:

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= \int \sin^2 x \cos^2 x \cdot \cos x\, dx \\ &= \int \sin^2 x\, (1 - \sin^2 x) \cos x\, dx \\ &= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx \\ &= \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x + C.\end{aligned}$$

Type II: $\int \tan^n x\, dx$, *or* $\int \cot^n x\, dx$, *where* n *is an integer.*

By use of the formulas

$$\tan^2 x = \sec^2 x - 1,$$
$$\cot^2 x = \csc^2 x - 1,$$

these integrals can be reduced to integrable forms.

Example:

$$\int \tan^4 x\, dx = \int \tan^2 x\, (\sec^2 x - 1)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$
$$= \tfrac{1}{3} \tan^3 x - \int (\sec^2 x - 1)\, dx$$
$$= \tfrac{1}{3} \tan^3 x - \tan x + x + C.$$

Type III: $\int \tan^m x \sec^n x\, dx$, *or* $\int \cot^m x \csc^n x\, dx$, *where n is a positive even integer.*

Example:

$$\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x\, (1 + \tan^2 x)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx + \int \tan^4 x \sec^2 x\, dx$$
$$= \tfrac{1}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \sin^3 x\, dx.$ *Ans.* $\frac{1}{3} \cos^3 x - \cos x + C.$

2. $\int \cos^3 x\, dx.$ *Ans.* $\sin x - \frac{1}{3} \sin^3 x + C.$

3. $\int \sin^2 x \cos^5 x\, dx.$ *Ans.* $\frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C.$

4. $\int \frac{\sin^3 x\, dx}{\cos^2 x}.$ *Ans.* $\sec x + \cos x + C.$

5. $\int \cos^4 x \sin^3 x\, dx.$ *Ans.* $-\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C.$

6. $\int \sin^5 2\theta\, d\theta$. *Ans.* $-\frac{1}{2}\cos 2\theta + \frac{1}{3}\cos^3 2\theta - \frac{1}{10}\cos^5 2\theta + C$.

7. $\int \sin^3\theta \cos^3\theta\, d\theta$. *Ans.* $-\frac{1}{8}\sin^2\theta\cos^4\theta - \frac{1}{12}\cos^4\theta + C$.

8. $\int \frac{\cos^3 x\, dx}{\sin^4 x}$. *Ans.* $\csc x - \frac{1}{3}\csc^3 x + C$.

9. $\int \sin^3 x \cos^2 x\, dx$. *Ans.* $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$.

10. $\int \cos^7\theta\, d\theta$. *Ans.* $\sin\theta - \sin^3\theta + \frac{3}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + C$.

11. $\int x \sin x \cos^2 x\, dx$. *Ans.* $-\frac{1}{3}x\cos^3 x + \frac{1}{3}\sin x - \frac{1}{9}\sin^3 x + C$.

12. $\int x \sin^3 x\, dx$. *Ans.* $\frac{1}{3}x\cos^3 x - x\cos x + \frac{2}{3}\sin x + \frac{1}{9}\sin^3 x + C$.

13. $\int \cot^4 x\, dx$. *Ans.* $-\frac{1}{3}\cot^3 x + \cot x + x + C$.

14. $\int \tan^3\theta\, d\theta$. *Ans.* $\frac{1}{2}\tan^2\theta + \log\cos\theta + C$.

15. $\int \cot^3\phi\, d\phi$. *Ans.* $-\frac{1}{2}\cot^2\phi - \log\sin\phi + C$.

16. $\int \tan^2\frac{\theta}{2}\, d\theta$. *Ans.* $2\tan\frac{\theta}{2} - \theta + C$.

17. $\int \tan^6 x\, dx$. *Ans.* $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$.

18. $\int \tan^5 x\, dx$. *Ans.* $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log\cos x + C$.

19. $\int \sec^4 x\, dx$. *Ans.* $\tan x + \frac{1}{3}\tan^3 x + C$.

20. $\int \csc^4 x\, dx$. *Ans.* $-\cot x - \frac{1}{3}\cot^3 x + C$.

21. $\int \tan^4 x \sec^4 x\, dx$. *Ans.* $\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$.

22. $\int \tan\theta \sec^4\theta\, d\theta$. *Ans.* $\frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta + C$.

23. $\int \csc^6 x\, dx$. *Ans.* $-\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$.

24. $\int x \sec^4 x\, dx$.
Ans. $x\tan x + \frac{1}{3}x\tan^3 x - \frac{1}{6}\tan^2 x + \frac{2}{3}\log\cos x + C$.

97. Transformation by multiple angles.

Type IV: $\int \sin^m x \cos^n x\, dx$, *where both* m *and* n *are positive even integers.*

In this case the integrand can always be transformed into an integrable expression in terms of sines and cosines of multiples of x, by means of the formulas

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$
$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

Examples:

$$(a) \qquad \int \sin^2 x\, dx = \tfrac{1}{2}\int (1 - \cos 2x)\, dx$$
$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C.$$

$$(b) \quad \int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx$$
$$= \tfrac{1}{8}\int (1 - \cos 4x)\, dx, \text{ etc.}$$

EXERCISES

Evaluate the following integrals.

1. $\int \cos^2 x\, dx.$ *Ans.* $\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.$

2. $\int \cos^4 x\, dx.$ *Ans.* $\tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$

3. $\int \sin^4 \theta \cos^2 \theta\, d\theta.$ *Ans.* $\tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C.$

4. $\int \sin^6 \theta\, d\theta.$ *Ans.* $\tfrac{5}{16}\theta - \tfrac{1}{4}\sin 2\theta + \tfrac{1}{48}\sin^3 2\theta + \tfrac{3}{64}\sin 4\theta + C.$

5. $\int \sin^4 x \cos^4 x\, dx.$ *Ans.* $\tfrac{3}{128}x - \tfrac{1}{128}\sin 4x + \tfrac{1}{1024}\sin 8x + C.$

6. $\int x \sin^2 x\, dx.$ *Ans.* $\tfrac{1}{4}x^2 - \tfrac{1}{4}x \sin 2x - \tfrac{1}{8}\cos 2x + C.$

7. $\int x \cos^2 x\, dx.$ *Ans.* $\tfrac{1}{4}x^2 + \tfrac{1}{4}x \sin 2x + \tfrac{1}{8}\cos 2x + C.$

8. $\int x \sin x \cos x\, dx.$ *Ans.* $-\tfrac{1}{4}x \cos 2x + \tfrac{1}{8}\sin 2x + C.$

9. $\int \theta \sin^2 \theta \cos^2 \theta\, d\theta.$ *Ans.* $\tfrac{1}{16}\theta^2 - \tfrac{1}{32}\theta \sin 4\theta - \tfrac{1}{128}\cos 4\theta + C.$

CHAPTER XV

INTEGRATION BY SUBSTITUTION

98. Integration by substitution. Many integrals may be evaluated by introducing a new variable of integration, say z, in place of the original variable x, the two variables being connected by some suitable formula. The change of variable is usually brought about by means of an explicit substitution

$$x = \phi(z),\ dx = \phi'(z)\, dz.$$

The process is called *integration by substitution*, and is highly important. It is to be remembered that *not merely* x, *but* dx *as well*, must be replaced by the proper value in terms of the new variable.

In general, the nature of the substitution to be made — *i.e.* the formula connecting x and z — must be determined by inspection of the given integrand. No general directions can be given; skill in the use of substitutions comes only with practice.

Examples: (*a*) Evaluate $\displaystyle\int \frac{\sqrt{x}\, dx}{1+x}$.

Let

$$\sqrt{x} = z,\ x = z^2,\ dx = 2\, z\, dz.$$

Then

$$\begin{aligned}\int \frac{\sqrt{x}\, dx}{1+x} &= 2\int \frac{z^2\, dz}{1+z^2} = 2\int\left(1 - \frac{1}{1+z^2}\right) dz \\ &= 2\, z - 2 \arctan z + C \\ &= 2\sqrt{x} - 2 \arctan \sqrt{x} + C.\end{aligned}$$

(*b*) Evaluate $\int \frac{x^2\,dx}{(x+1)^2}$.

Let $x + 1 = z, \quad x = z - 1, \quad dx = dz.$

Then

$$\int \frac{x^2\,dx}{(x+1)^2} = \int \frac{z^2 - 2z + 1}{z^2}\,dz = \int \left(1 - \frac{2}{z} + \frac{1}{z^2}\right) dz$$

$$= z - 2\log z - \frac{1}{z} + C$$

$$= x + 1 - 2\log(x+1) - \frac{1}{x+1} + C$$

$$= x - 2\log(x+1) - \frac{1}{x+1} + C'.$$

(*c*) Evaluate $\int \frac{(e^x - 1)e^x\,dx}{e^x + 1}$.

Let

$$e^x + 1 = z, \quad e^x\,dx = dz.$$

Then

$$\int \frac{(e^x - 1)e^x\,dx}{e^x + 1} = \int \frac{(z-2)\,dz}{z} = z - 2\log z + C$$

$$= e^x - 2\log(e^x + 1) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{1 - \sqrt{x}}$. *Ans.* $-2\sqrt{x} - 2\log(1 - \sqrt{x}) + C$.

2. $\int \frac{x+3}{\sqrt{1+2x}}\,dx$. *Ans.* $\frac{1}{3}(x+8)\sqrt{1+2x} + C$.

3. $\int \frac{x^2\,dx}{(x+1)^3}$. *Ans.* $\frac{4x+3}{2(x+1)^2} + \log(x+1) + C$.

4. $\int x^3 e^{x^2}\,dx$. *Ans.* $\frac{1}{2}e^{x^2}(x^2 - 1) + C$.

5. $\int \frac{dx}{1 + \sqrt{1+x}}$. *Ans.* $2\sqrt{1+x} - 2\log(1 + \sqrt{1+x}) + C$.

6. $\int \frac{x\,dx}{1 + \sqrt{x}}$. *Ans.* $\frac{2}{3}x^{\frac{3}{2}} - x + 2\sqrt{x} - 2\log(1 + \sqrt{x}) + C$.

7. $\int \sin\sqrt{x}\,dx$. *Ans.* $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$.

8. $\int x^5 e^{-x^3}\,dx.$ *Ans.* $-\frac{1}{3}x^3e^{-x^3} - \frac{1}{3}e^{-x^3} + C.$

9. $\int \frac{dx}{x\sqrt{x^2 - a^2}}.$ *Ans.* $\frac{1}{a}\arctan\frac{\sqrt{x^2 - a^2}}{a} + C.$

10. $\int x^3\sqrt{a^2 - x^2}\,dx.$ *Ans.* $-\frac{1}{3}a^2(a^2 - x^2)^{\frac{3}{2}} + \frac{1}{5}(a^2 - x^2)^{\frac{5}{2}} + C.$

11. $\int \frac{\sqrt{x^2 - a^2}\,dx}{x}.$ *Ans.* $\sqrt{x^2 - a^2} + a\arcsin\frac{a}{x} + C.$

12. $\int \frac{x^3dx}{\sqrt{a^2 + x^2}}.$ *Ans.* $\frac{1}{3}(a^2 + x^2)^{\frac{3}{2}} - a^2(a^2 + x^2)^{\frac{1}{2}} + C.$

13. $\int \frac{x^3\,dx}{(a^2 + x^2)^2}.$ *Ans.* $\frac{a^2}{2(a^2 + x^2)} + \frac{1}{2}\log(a^2 + x^2) + C.$

14. $\int \frac{\sqrt{x^3 - 1}}{x}\,dx.$ *Ans.* $\frac{2}{3}\sqrt{x^3 - 1} - \frac{2}{3}\arctan\sqrt{x^3 - 1} + C.$

15. $\int \frac{x^3\,dx}{a^2 - x^2}.$ *Ans.* $\frac{1}{2}(a^2 - x^2) - \frac{1}{2}a^2\log(a^2 - x^2) + C.$

16. $\int \frac{(x^2 - a^2)^{\frac{3}{2}}}{x}\,dx.$

Ans. $\frac{1}{3}(x^2 - a^2)^{\frac{3}{2}} - a^2(x^2 - a^2)^{\frac{1}{2}} - a^3\arcsin\frac{a}{x} + C.$

17. $\int \frac{\sin x\cos x\,dx}{1 + \sin x}.$ *Ans.* $\sin x - \log(1 + \sin x) + C.$

18. $\int \frac{e^{2x}dx}{\sqrt{1 + e^x}}.$ *Ans.* $\frac{2}{3}(e^x - 2)\sqrt{1 + e^x} + C.$

19. $\int \frac{\sec^2\theta\tan^2\theta\,d\theta}{\sqrt{1 + \tan\theta}}.$

Ans. $\frac{2}{5}(1 + \tan\theta)^{\frac{5}{2}} - \frac{4}{3}(1 + \tan\theta)^{\frac{3}{2}} + 2(1 + \tan\theta)^{\frac{1}{2}} + C.$

Evaluate the following integrals by the substitution $x = \frac{a}{z}$.

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}}.$ *Ans.* $-\frac{1}{a}\arcsin\frac{a}{x} + C.$

21. $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}.$ *Ans.* $\frac{x}{a^2\sqrt{a^2 + x^2}} + C.$

22. $\int \frac{\sqrt{a^2 - x^2}\,dx}{x^4}.$ *Ans.* $-\frac{(a^2 - x^2)^{\frac{3}{2}}}{3\,a^2x^3} + C.$

23. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$ *Ans.* $-\frac{\sqrt{a^2 - x^2}}{a^2x} + C.$

24. $\int \frac{dx}{x^2(a^2 + x^2)}.$ *Ans.* $-\frac{1}{a^2x} + \frac{1}{a^3}\arctan\frac{a}{x} + C.$

99. Trigonometric substitutions. Many integrals can be evaluated by substituting a trigonometric function for x. The following trigonometric substitutions are especially useful:

When the integrand involves $a^2 - x^2$, try $x = a \sin \theta$.
When the integrand involves $a^2 + x^2$, try $x = a \tan \theta$.
When the integrand involves $x^2 - a^2$, try $x = a \sec \theta$.

Example: Evaluate $\displaystyle\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}}$.

Putting $x = a \sin \theta$, $dx = a \cos \theta \, d\theta$, we get

$$\int \frac{a \cos \theta \, d\theta}{(a^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}} = \frac{1}{a^2} \int \frac{\cos \theta \, d\theta}{(1 - \sin^2 \theta)^{\frac{3}{2}}} = \frac{1}{a^2} \int \frac{\cos \theta \, d\theta}{\cos^3 \theta}.$$

$$= \frac{1}{a^2} \int \sec^2 \theta \, d\theta = \frac{1}{a^2} \tan \theta + C.$$

Since $\sin \theta = \dfrac{x}{a}$, it follows that $\tan \theta = \dfrac{x}{\sqrt{a^2 - x^2}}$, whence

$$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C.$$

EXERCISES

Evaluate the following integrals by a trigonometric substitution.

1. $\displaystyle\int \frac{dx}{x^2 \sqrt{a^2 - x^2}}$. *Ans.* $\displaystyle -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$

2. $\displaystyle\int \frac{\sqrt{a^2 - x^2}}{x^2} dx$. *Ans.* $\displaystyle -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin \frac{x}{a} + C.$

3. $\displaystyle\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$. *Ans.* $\displaystyle \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$

4. $\displaystyle\int \frac{dx}{x(1 + x^2)}$. *Ans.* $\displaystyle \frac{1}{2} \log \frac{x^2}{1 + x^2} + C.$

5. $\displaystyle\int \frac{dx}{x \sqrt{x^2 - a^2}}$. *Ans.* $\displaystyle -\frac{1}{a} \arcsin \frac{a}{x} + C.$

6. $\displaystyle\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{3}{2}}}$. *Ans.* $\displaystyle \frac{x}{\sqrt{a^2 - x^2}} - \arcsin \frac{x}{a} + C.$

7. $\displaystyle\int \frac{dx}{x^2 \sqrt{a^2 + x^2}}$. *Ans.* $\displaystyle -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C.$

8. $\int \frac{\sqrt{x^2 - a^2}}{x}\,dx.$ *Ans.* $\sqrt{x^2 - a^2} + a \arcsin \frac{a}{x} + C.$

9. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}.$ *Ans.* $\frac{-x}{a^2\sqrt{x^2 - a^2}} + C.$

10. $\int \frac{dx}{x^2\sqrt{4x^2 - 1}}.$ *Ans.* $\frac{\sqrt{4x^2 - 1}}{x} + C.$

11. $\int \frac{dx}{x(x^2 - a^2)}.$ *Ans.* $\frac{1}{2a^2} \log \frac{x^2 - a^2}{x^2} + C.$

12. $\int \frac{dx}{x\sqrt{2ax - x^2}}.$ (Put $x = 2a \sin^2 \theta$.) *Ans.* $-\frac{\sqrt{2ax - x^2}}{ax} + C.$

13. $\int \frac{dx}{x\sqrt{2ax + x^2}}.$ (Put $x = 2a \tan^2 \theta$.) *Ans.* $-\frac{\sqrt{2ax + x^2}}{ax} + C.$

14. $\int \frac{dx}{x\sqrt{x^2 - 2ax}}.$ (Put $x = 2a \sec^2 \theta$.) *Ans.* $\frac{\sqrt{x^2 - 2ax}}{ax} + C.$

15. $\int \frac{dx}{x^2\sqrt{2ax - x^2}}.$

Ans. $-\frac{1}{2a^2}\left(\frac{\sqrt{2ax - x^2}}{x} + \frac{(2ax - x^2)^{\frac{3}{2}}}{3x^3}\right) + C.$

16. $\int \frac{dx}{x^2\sqrt{x^2 - 2ax}}.$ *Ans.* $\frac{1}{2a^2}\left(\frac{\sqrt{x^2 - 2ax}}{x} - \frac{(x^2 - 2ax)^{\frac{3}{2}}}{3x^3}\right) + C.$

17. $\int \sqrt{a^2 - x^2}\,dx.$ *Ans.* $\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \arcsin \frac{x}{a} + C.$

18. $\int \frac{dx}{x\sqrt{a^2 - x^2}}.$ *Ans.* $\frac{1}{2a} \log \frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} + C.$

CHAPTER XVI

INTEGRATION OF RATIONAL FRACTIONS

100. Preliminary step. We take up next the problem of integrating a *rational algebraic fraction — i.e.* the quotient of two polynomials.

As noted in § 91, the first step in dealing with an integral of this type is to *carry out the indicated division until the numerator is of lower degree than the denominator.*

101. Partial fractions. If the denominator is of the first degree, or of the second degree with complex roots, the integration can be performed by the fundamental formulas. Many examples have already arisen; *e.g.* Exs. 24, 33, 35, 40, p. 147. In other cases the integral may sometimes be evaluated either directly or after a suitable substitution, as in Exs. 27, 43, 47, p. 147, or 11, p. 156.

In general, however, we must resort to the method of "partial fractions." The first step (after the numerator has been made of lower degree than the denominator) is to resolve the denominator into real linear or quadratic factors*. If this can be done, the given fraction can then be expressed as a sum of *partial fractions* whose denominators are the factors of the original denominator. We proceed to show how to do this.

* The term "quadratic factor" means here a factor of the second degree whose linear factors are imaginary; *i.e.* a factor of the form $ax^2 + bx + c$ with $b^2 - 4ac < 0$.

102. Distinct linear factors. The simplest case is that in which the denominator can be broken up into real linear factors, none of which are repeated.

Example: Evaluate $\int \frac{x^3+2}{x^3-x}\,dx$.

By division we find

$$\frac{x^3+2}{x^3-x} = 1 + \frac{x+2}{x^3-x}.$$

The factors of the denominator are x, $x+1$, $x-1$. Assume

$$\frac{x+2}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

where A, B, C are constants to be determined. Clearing of fractions, we find

$$x + 2 = A(x^2-1) + Bx(x-1) + Cx(x+1).$$

This relation must hold for all values of x. Hence,

putting $x = 0$, we find $A = -2$,
$x = -1$, $B = \frac{1}{2}$,
$x = 1$, $C = \frac{3}{2}$.

Thus

$$\int \frac{x^3+2}{x(x^2-1)}\,dx = \int \left(1 - \frac{2}{x} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{3}{2}\cdot\frac{1}{x-1}\right)dx$$
$$= x - 2\log x + \tfrac{1}{2}\log(x+1) + \tfrac{3}{2}\log(x-1) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{a^2-x^2}$. *Ans.* $\frac{1}{2a}\log\frac{a+x}{a-x} + C$.
2. $\int \frac{x^2\,dx}{x^2-4}$. *Ans.* $x - \log\frac{x+2}{x-2} + C$.
3. $\int \frac{dx}{x^2-x}$. *Ans.* $\log\frac{x-1}{x} + C$.
4. $\int \frac{x^3\,dx}{x^2+3x+2}$. *Ans.* $\frac{1}{2}x^2 - 3x - \log(x+1) + 8\log(x+2) + C$.
5. $\int \frac{(x^2-3x)\,dx}{x^2+7x+6}$. *Ans.* $x + \frac{4}{5}\log(x+1) - \frac{54}{5}\log(x+6) + C$.

6. $\int \frac{u^4\,du}{3u^2 - 2u - 1}$.

Ans. $\frac{1}{9}u^3 + \frac{1}{9}u^2 + \frac{7}{27}u - \frac{1}{32}\frac{1}{4}\log(3u + 1) + \frac{1}{4}\log(u - 1) + C$.

7. $\int \frac{dx}{x(1 - x^2)}$. *Ans.* $\frac{1}{2}\log\frac{x^2}{1 - x^2} + C$.

8. $\int \frac{(2x^3 + x - 1)\,dx}{x^3 + x^2 - 4x - 4}$.

Ans. $2x + \frac{17}{12}\log(x - 2) - \frac{19}{4}\log(x + 2) + \frac{4}{3}\log(x + 1) + C$.

9. $\int \frac{x^4\,dx}{x^3 + 2x^2 - x - 2}$.

Ans. $\frac{1}{2}x^2 - 2x + \frac{1}{8}\log(x - 1) - \frac{1}{2}\log(x + 1) + \frac{16}{3}\log(x + 2) + C$.

10. $\int \frac{x\,dx}{1 - x^4}$. *Ans.* $\frac{1}{4}\log\frac{1 + x^2}{1 - x^2} + C$.

11. $\int \frac{x\,dx}{x^4 + 4x^2 + 3}$. *Ans.* $\frac{1}{4}\log\frac{x^2 + 1}{x^2 + 3} + C$.

12. $\int \frac{e^{3x}\,dx}{1 - e^{2x}}$. *Ans.* $-e^x + \frac{1}{2}\log\frac{1 + e^x}{1 - e^x} + C$.

13. $\int \frac{\sin\theta\,d\theta}{3\cos\theta - \cos^2\theta}$. *Ans.* $\frac{1}{3}\log(1 - 3\sec\theta) + C$.

14. $\int \frac{dx}{x\sqrt{a^2 - x^2}}$. *Ans.* $\frac{1}{2a}\log\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} + C$.

15. $\int \frac{dx}{x\sqrt{a^2 + x^2}}$. *Ans.* $\frac{1}{2a}\log\frac{a - \sqrt{a^2 + x^2}}{a + \sqrt{a^2 + x^2}} + C$.

16. $\int \frac{\sqrt{a^2 - x^2}\,dx}{x}$. *Ans.* $\sqrt{a^2 - x^2} + \frac{1}{2}a\log\frac{\sqrt{a^2 - x^2} - a}{\sqrt{a^2 - x^2} + a} + C$.

17. $\int \frac{\sqrt{a^2 + x^2}\,dx}{x}$. *Ans.* $\sqrt{a^2 + x^2} + \frac{1}{2}a\log\frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} + C$.

18. $\int \csc\theta\,d\theta$. $\left(\text{Note: } \csc\theta = \frac{1}{\sin\theta} = \frac{\sin\theta}{\sin^2\theta} = \frac{\sin\theta}{1 - \cos^2\theta}.\right)$

Ans. $\frac{1}{2}\log\frac{1 - \cos\theta}{1 + \cos\theta} + C$.

19. $\int \sec\theta\,d\theta$. *Ans.* $\frac{1}{2}\log\frac{1 + \sin\theta}{1 - \sin\theta} + C$.

20. $\int \frac{dx}{\sqrt{a^2 + x^2}}$. *Ans.* $\log(x + \sqrt{a^2 + x^2}) + C$.

21. $\int \frac{dx}{\sqrt{x^2 - a^2}}$. *Ans.* $\log(x + \sqrt{x^2 - a^2}) + C$.

22. $\int \frac{dx}{\sqrt{2ax + x^2}}$. *Ans.* $\log(a + x + \sqrt{2ax + x^2}) + C$.

103. Repeated linear factors. If the denominator contains a factor $(x - \alpha)^r$, the above method fails, since there would be r partial fractions with denominator $x - \alpha$, and these could be combined into a single one. We can obtain the desired result in this case by assuming, corresponding to the factor $(x - \alpha)^r$, r partial fractions

$$\frac{A}{(x-\alpha)^r} + \frac{B}{(x-\alpha)^{r-1}} + \cdots + \frac{D}{x-\alpha}.$$

Example: Evaluate $\int \frac{x^3 - 1}{x(x+1)^3}\,dx$.

Assume

$$\frac{x^3-1}{x(x+1)^3} = \frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1},$$

$$x^3 - 1 = A(x+1)^3 + Bx + Cx(x+1) + Dx(x+1)^2.$$

Put $x = 0$: $A = -1$,

$x = -1$: $B = 2$.

To find C and D, we may assign any two other values to x, say $x = 1$ and $x = 2$, thus obtaining two simultaneous equations to solve for C and D; or we may equate the coefficients of like powers of x in the two members of the identity.

Equate coefficients of x^3: $A + D = 1$, $D = 2$.

Equate coefficients of x^2: $3A + C + 2D = 0$, $C = -1$.

Whence

$$\int \frac{x^3-1}{x(x+1)^3}\,dx = \int\left(-\frac{1}{x} + \frac{2}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{2}{x+1}\right)dx$$

$$= -\log x - \frac{1}{(x+1)^2} + \frac{1}{x+1} + 2\log(x+1) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{x^3 + x^2}$. *Ans.* $-\frac{1}{x} + \log\frac{1+x}{x} + C$.

2. $\int \frac{(x^2 + 3x - 1)\,dx}{(x+1)^2}$. *Ans.* $x + \log(x+1) + \frac{3}{x+1} + C$.

3. $\int \frac{(x^3-1)\,dx}{x(x-2)^2}$.

Ans. $x - \frac{1}{4}\log x + \frac{17}{4}\log(x-2) - \frac{7}{2(x-2)} + C$.

4. $\int \frac{dx}{x^2(1-x^2)}$. *Ans.* $-\frac{1}{x} + \frac{1}{2}\log\frac{1+x}{1-x} + C$.

5. $\int \frac{x\,dx}{(1-x)^5}$.

6. $\int \frac{dx}{(1-x)^3}$.

7. $\int \frac{x^4 - 2x^2 + 3x + 1}{(x^2-1)^2}\,dx$.

8. $\int \frac{x^2 - 5x + 6}{(x-1)^2}\,dx$.

9. $\int \frac{dx}{(x^2+x)(x-1)^2}$.

Ans. $\log x - \frac{1}{4}\log(x+1) - \frac{3}{4}\log(x-1) - \frac{1}{2(x-1)} + C$.

10. $\int \frac{x^2\,dx}{x^4 + 12x^3 + 52x^2 + 96x + 64}$.

Ans. $\frac{-5x-12}{x^2+6x+8} + 2\log\frac{x+4}{x+2} + C$.

11. $\int \frac{dx}{e^{2x} - 2e^x}$. *Ans.* $\frac{1}{2}e^{-x} - \frac{x}{4} + \frac{1}{4}\log(e^x - 2) + C$.

12. $\int \frac{dx}{(1+e^x)^2}$. *Ans.* $\frac{1}{1+e^x} + x - \log(1+e^x) + C$.

13. $\int \frac{d\theta}{\cos\theta\sin^2\theta}$. *Ans.* $-\csc\theta + \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} + C$.

14. $\int \frac{dx}{(a^2-x^2)^2}$. *Ans.* $\frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\log\frac{a+x}{a-x} + C$.

15. $\int \frac{dx}{x^3\sqrt{a^2-x^2}}$. *Ans.* $-\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{4a^3}\log\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + C$.

16. $\int \frac{dx}{x(a^2+x^2)^2}$. *Ans.* $\frac{1}{2a^2(a^2+x^2)} + \frac{1}{2a^4}\log\frac{x^2}{a^2+x^2} + C$.

17. $\int \frac{dx}{x(a^2-x^2)^{\frac{3}{2}}}$. *Ans.* $\frac{1}{a^2\sqrt{a^2-x^2}} + \frac{1}{2a^3}\log\frac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a} + C$.

18. $\int \sec^3\theta\,d\theta$. (Note: $\sec^3\theta = \frac{1}{\cos^3\theta} = \frac{\cos\theta}{\cos^4\theta} = \frac{\cos\theta}{(1-\sin^2\theta)^2}$.)

Ans. $\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\log\frac{1+\sin\theta}{1-\sin\theta} + C$.

19. $\int \csc^3\theta\,d\theta$. *Ans.* $-\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{4}\log\frac{1-\cos\theta}{1+\cos\theta} + C$.

20. $\int \sqrt{a^2+x^2}\,dx$.

Ans. $\frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\log(x + \sqrt{a^2+x^2}) + C$.

104. Quadratic factors. Corresponding to a factor in the denominator of the form * $ax^2 + bx + c$ where $b^2 - 4ac < 0$, we assume the terms $\dfrac{A(2ax + b)}{ax^2 + bx + c} + \dfrac{B}{ax^2 + bx + c}$.

The case in which the denominator has repeated quadratic factors is of less importance, and will be omitted.

Example: Evaluate $\displaystyle\int \frac{x^2 + 4x + 10}{x^3 + 2x^2 + 5x}\,dx$.

Assume

$$\frac{x^2 + 4x + 10}{x^3 + 2x^2 + 5x} = \frac{A}{x} + \frac{B(2x + 2)}{x^2 + 2x + 5} + \frac{C}{x^2 + 2x + 5},$$

$$x^2 + 4x + 10 = A(x^2 + 2x + 5) + Bx(2x + 2) + Cx.$$

Put $x = 0$: $5A = 10$, $A = 2$.

Equate coefficients of x^2: $A + 2B = 1$, $B = -\frac{1}{2}$.

Equate coefficients of x: $2A + 2B + C = 4$, $C = 1$.

Whence

$$\int \frac{x^2 + 4x + 10}{x^3 + 2x^2 + 5x}\,dx$$

$$= \int \left(\frac{2}{x} - \frac{1}{2} \cdot \frac{2x + 2}{x^2 + 2x + 5} + \frac{1}{x^2 + 2x + 5}\right) dx$$

$$= 2 \log x - \frac{1}{2} \log (x^2 + 2x + 5) + \frac{1}{2} \arctan \frac{x + 1}{2} + C.$$

EXERCISES

Evaluate the following integrals.

1. $\displaystyle\int \frac{x^2\,dx}{x^2 - 4x + 5}$.

Ans. $x + 2 \log (x^2 - 4x + 5) + 3 \arctan (x - 2) + C$.

2. $\displaystyle\int \frac{x\,dx}{4x^2 - 4x + 3}$.

Ans. $\frac{1}{8} \log (4x^2 - 4x + 3) + \frac{1}{4\sqrt{2}} \arctan \frac{2x - 1}{\sqrt{2}} + C$.

* Of course such a factor might be broken up into complex linear factors, after which the process of § 102 would apply. The present method has the advantage of avoiding imaginaries.

3. $\int \frac{x^3\,dx}{x^2 - 2x + 10}$.

Ans. $\frac{1}{2}x^2 + 2x - 3\log(x^2 - 2x + 10) - \frac{26}{3}\arctan\frac{x-1}{3} + C.$

4. $\int \frac{dx}{x^3 + 4x^2 + 8x}$.

Ans. $\frac{1}{16}\log\frac{x^2}{x^2 + 4x + 8} - \frac{1}{8}\arctan\frac{x+2}{2} + C.$

5. $\int \frac{x\,dx}{1 + x^4}$.

6. $\int \frac{(2x - 1)\,dx}{3x^2 - 3x + 4}$.

7. $\int \frac{x^2\,dx}{a^4 - x^4}$.

Ans. $\frac{1}{4a}\log\frac{a + x}{a - x} - \frac{1}{2a}\arctan\frac{x}{a} + C.$

8. $\int \frac{dx}{x^2(1 + x^2)}$.

Ans. $-\frac{1}{x} - \arctan x + C.$

9. $\int \frac{x\,dx}{x^3 + x^2 + 4x + 4}$.

Ans. $\frac{1}{10}\log\frac{x^2 + 4}{(x + 1)^2} + \frac{2}{5}\arctan\frac{x}{2} + C.$

10. $\int \frac{x\,dx}{1 - x^8}$.

Ans. $\frac{1}{8}\log\frac{1 + x^2}{1 - x^2} + \frac{1}{4}\arctan x^2 + C$

11. $\int \frac{x^3\,dx}{(1 + x^2)^2}$.

Ans. $\frac{1}{2}\log(1 + x^2) + \frac{1}{2(1 + x^2)} + C.$

12. $\int \frac{\arctan x}{x^3}\,dx$.

Ans. $-\frac{\arctan x}{2x^2} - \frac{1}{2x} - \frac{1}{2}\arctan x + C.$

13. $\int \frac{\arctan x\,dx}{x^2(1 + x^2)}$.

Ans. $-\frac{1}{x}\arctan x - \frac{1}{2}\arctan^2 x + \frac{1}{2}\log\frac{x^2}{1 + x^2} + C.$

14. $\int \frac{\cos\theta\,d\theta}{\sin\theta + \sin^3\theta}$

Ans. $\frac{1}{2}\log\frac{\sin^2\theta}{1 + \sin^2\theta} + C.$

MISCELLANEOUS EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{(a + bx)^2}$.

2. $\int \frac{x\,dx}{a + bx}$.

3. $\int \frac{x\,dx}{(a + x)^2}$.

4. $\int \frac{x^2\,dx}{(a + x)^2}$.

5. $\int \frac{x\,dx}{(a + x)^2(b + x)}$.

6. $\int \frac{x\,dx}{(a^2 + x^2)^2}$.

7. $\int \frac{x^3 - 1}{x + x^2}\,dx$.

8. $\int \frac{x^2\,dx}{x^6 - 1}$.

9. $\int \frac{dx}{x(1-x^3)}$.

10. $\int \frac{x\,dx}{x^4-4}$.

11. $\int \frac{x^3\,dx}{a^4-x^4}$.

12. $\int \frac{x\,dx}{2x^2+8x+1}$.

13. $\int \frac{\sqrt{x}\,dx}{1-x^2}$.

14. $\int \frac{x\,dx}{(2-x)^6}$.

15. $\int \frac{dx}{1-e^x}$.

16. $\int \frac{d\theta}{\tan\theta-\cot\theta}$.

17. $\int \frac{(x^2-2)\,dx}{1+6x-x^3}$.

18. $\int \frac{d\theta}{\sin\theta\cos\theta}$.

19. $\int \frac{d\theta}{\sin\theta\cos^2\theta}$.

20. $\int \frac{\sin^2\theta\,d\theta}{\cos\theta}$.

21. $\int \frac{dx}{x\sqrt{1-x}}$.

22. $\int \frac{dx}{x^2\sqrt{1-x}}$.

23. $\int \frac{x\,dx}{(1+x^2)^3}$.

24. $\int \frac{(a^2+x^2)^{\frac{3}{2}}}{x}\,dx$.

25. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

26. $\int \frac{x^2\,dx}{\sqrt{a^2+x^2}}$.

27. $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}}$.

28. $\int \frac{x^2\,dx}{(a^2+x^2)^{\frac{3}{2}}}$.

29. $\int \frac{dx}{x(1+x^3)}$.

30. $\int \frac{x\,dx}{\sqrt{2ax-x^2}}$.

CHAPTER XVII

THE DEFINITE INTEGRAL

105. The definite integral. Let $f(x)$ be a given function, $F(x)$ an integral of $f(x)$, and $x = a$ and $x = b$ two given values of x. The *change in the value of the integral* $F(x)$ as x changes from a to b, *i.e.* the quantity $F(b) - F(a)$, is called the *definite integral* * *of* $f(x)$ *between the "limits" a and b*, or simply the definite integral *from a to b*, and is denoted by the symbol $\int_a^b f(x)dx$. It is called the *definite* integral because its value is independent of the constant of integration.

To evaluate a definite integral, we have merely to find the indefinite integral, and then *subtract its value at the "lower limit" a from its value at the "upper limit" b*. It is customary to use the symbol $F(x)\Big]_a^b$ as meaning $F(b) - F(a)$. Thus

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a).$$

Since the constant of integration disappears, there is no object in writing it at all.

Examples: (*a*) $\int_0^1 (x+1)^2\,dx = \frac{(x+1)^3}{3}\Big]_0^1 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$

(*b*†) $\int_1^e \log x\,dx = x \log x\Big]_1^e - \int_1^e dx = e - (e-1) = 1.$

*Provided $f(x)$ satisfies certain general conditions whose nature will be discussed in § 110.

† Cf. example (*b*), § 94.

EXERCISES

Evaluate the following definite integrals.*

1. $\int_{-2}^{2} x^2(x+2)\,dx$. *Ans.* $\frac{32}{3}$.

2. $\int_{-1}^{0} \frac{dx}{1-x}$. *Ans.* $\log 2$.

3. $\int_{0}^{\pi} \cos\frac{\theta}{2}\,d\vartheta$. *Ans.* 2.

4. $\int_{0}^{2} \frac{x\,dx}{4+x^2}$. *Ans.* $\frac{1}{2}\log 2$.

5. $\int_{\frac{2\pi}{3}}^{\pi} \sin\phi\,d\phi$. *Ans.* $\frac{1}{2}$.

6. $\int_{-1}^{1} (x^2+2x+2)\,dx$. *Ans.* $\frac{14}{3}$.

7. $\int_{-1}^{0} \frac{dx}{1+x^2}$. *Ans.* $\frac{\pi}{4}$.

8. $\int_{-a}^{0} \frac{dx}{\sqrt{a^2-x^2}}$. *Ans.* $\frac{\pi}{2}$.

9. $\int_{1}^{e^2} \frac{dx}{x}$. *Ans.* 2.

10. $\int_{0}^{2} \frac{x^3\,dx}{x+1}$. *Ans.* 1.568.

11. $\int_{1}^{2} ye^y\,dy$. *Ans.* e^2.

12. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{1+\sin^2\theta}$. *Ans.* $\frac{\pi}{2}$.

13. $\int_{0}^{\log 3} e^{2z}\,dz$. *Ans.* 4.

14. $\int_{0}^{1} \arcsin x\,dx$. *Ans.* $\frac{\pi}{2}-1$.

15. $\int_{0}^{\frac{\pi}{2}} \sin^2\theta\,d\theta$. *Ans.* $\frac{\pi}{4}$.

16. $\int_{0}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$. *Ans.* $\frac{\pi}{4}$.

17. $\int_{0}^{\frac{\pi}{2}} x\sin x\,dx$. *Ans.* 1.

18. $\int_{0}^{\frac{\pi}{6}} \tan 2\theta\,d\theta$. *Ans.* 0.347.

19. $\int_{0}^{1} x\sinh x\,dx$. *Ans.* $\frac{1}{e}$.

20. $\int_{0}^{\pi} \cos^3\phi\,d\phi$. *Ans.* 0.

21. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{1+4x^2}$. *Ans.* $\frac{\pi}{4}$.

22. $\int_{0}^{1} \frac{x^2-2}{x+1}\,dx$. *Ans.* -1.193.

23. $\int_{0}^{\frac{\pi}{2}} x\cos 2x\,dx$. *Ans.* $-\frac{1}{2}$.

24. $\int_{0}^{\log 2} \frac{e^x\,dx}{5e^x-1}$. *Ans.* 0.162.

106. Interchanging limits. The effect of interchanging the limits in a definite integral is to change the sign of the integral. For,

$$\int_a^b f(x)\,dx = F(b) - F(a) = -\int_b^a f(x)\,dx.$$

* To save time, it is recommended that from this point on, the more difficult integrals be obtained from the Table, pp. 361–363.

107. Change of limits corresponding to a change of variable. In the definite integral $\int_a^b f(x)\,dx$ it is of course implied that a and b are the limiting values of the variable of integration x. If we change the variable by a substitution

$$x = \phi(z),$$

we must either *return to the original variable* before substituting the limits, or *change the limits to correspond with the change of variable*. The latter method is usually preferable.

The new limits are found, of course, from the equation of substitution

$$x = \phi(z).$$

Example: Evaluate $\int_{-1}^{1} x\sqrt{1-x}\,dx$.

Let

$$1 - x = z, \quad x = 1 - z, \quad dx = -dz.$$

When

$$x = -1, \qquad z = 2;$$

when

$$x = 1, \qquad z = 0.$$

Hence

$$\int_{-1}^{1} x\sqrt{1-x}\,dx = -\int_2^0 (1-z)z^{\frac{1}{2}}\,dz$$

$$= \int_2^0 (z^{\frac{3}{2}} - z^{\frac{1}{2}})dz = \tfrac{2}{5}z^{\frac{5}{2}} - \tfrac{2}{3}z^{\frac{3}{2}}\Big]_2^0 = -\tfrac{4}{15}\sqrt{2}.$$

EXERCISES

Evaluate the following integrals by substitution.

1. $\int_1^6 \frac{x\,dx}{\sqrt{3+x}}$. *Ans.* $\frac{20}{3}$.
2. $\int_1^2 x^3\sqrt{x^2-1}\,dx$. *Ans.* $\frac{14}{5}\sqrt{3}$.
3. $\int_0^4 \frac{dx}{1+\sqrt{x}}$. *Ans.* 1.803.
4. $\int_0^a \frac{x^3\,dx}{(a^2+x^2)^{\frac{3}{2}}}$. *Ans.* $(\frac{3}{2}\sqrt{2}-2)a$.
5. $\int_0^1 \frac{x\,dx}{(1+x)^4}$. *Ans.* $\frac{1}{12}$.
6. $\int_2^3 \frac{x\,dx}{1-x^4}$. *Ans.* -0.072.
7. $\int_{\frac{a}{2}}^{a} \frac{\sqrt{a^2-x^2}}{x^2}\,dx$. *Ans.* $\sqrt{3} - \frac{\pi}{3}$.

8. $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx.$ *Ans.* $\frac{1}{2}\pi a^2.$

9. $\int_{0}^{a} \frac{x^2\, dx}{(4a^2 - x^2)^{\frac{3}{2}}}.$ *Ans.* $\frac{1}{3}\sqrt{3} - \frac{\pi}{6}.$

10. $\int_{1}^{2} \frac{dx}{x(x^2 + 1)}.$ *Ans.* $\frac{1}{2}\log\frac{8}{5}.$

11. $\int_{a}^{2a} \frac{dx}{x^2\sqrt{a^2 + x^2}}.$ *Ans.* $\frac{1}{a^2}\left(\sqrt{2} - \frac{1}{2}\sqrt{5}\right).$

12. $\int_{0}^{a} \frac{dx}{\sqrt{a^2 + x^2}}.$ *Ans.* $\log(1 + \sqrt{2}).$

108. Integrand containing two or more variables. We shall have frequent occasion to evaluate definite integrals of the form $\int_a^b y\, dx$, $\int_a^b y^2\, dx$, $\int_a^b xy\, dx$, etc., where the auxiliary variable y is defined as a function of the variable of integration x. To do this, we have merely to replace y in the integrand by its value in terms of x, as in example (*a*) below. Frequently, however, it is more convenient to substitute for x and dx their values in terms of y and dy, and change the limits, as in example (*b*).

Examples: (*a*) Given

(1) $$x^2 + y^2 = a^2,$$

evaluate $\int_0^a xy\, dx.$

We have

$$\int_0^a xy\, dx = \int_0^a x\sqrt{a^2 - x^2}\, dx$$

$$= -\tfrac{1}{2}\int_0^a \sqrt{a^2 - x^2} \cdot -2x\, dx$$

$$= -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\Big]_0^a = \frac{a^3}{3}.$$

(*b*) A better method:

From (1), we find

$$2x\, dx + 2y\, dy = 0,$$

whence

$$x\, dx = -y\, dy.$$

Also, when $x = 0$, $y = a$; when $x = a$, $y = 0$.
Hence

$$\int_0^a xy\,dx = -\int_a^0 y^2\,dy = \frac{a^3}{3}.$$

109. Parametric equations. When x and y are both given in terms of a third variable, or parameter, integrals of the types discussed in § 108 are evaluated by substituting for x, y, and dx their values in terms of the parameter, and changing the limits.

Example: Given $x = a\cos\phi$, $y = b\sin\phi$, evaluate $\int_0^a y\,dx$.

Differentiating the first of the given equations, we find

$$dx = -a\sin\phi\,d\phi.$$

Also, from the same equation, when $x = 0$, $\phi = \frac{1}{2}\pi$; when $x = a$, $\phi = 0$. Hence

$$\int_0^a y\,dx = -ab\int_{\frac{\pi}{2}}^0 \sin^2\phi\,d\phi$$

$$= -ab\left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_{\frac{\pi}{2}}^0 = \frac{\pi ab}{4}.$$

EXERCISES

Evaluate the following integrals.

1. $\int_1^{e^2} y\,dx$, where $y = \log x$. *Ans.* $e^2 + 1$.

2. $\int_0^a y\,dx$, where $y^2 = 4ax$. *Ans.* $\frac{4}{3}a^2$.

3. $\int_0^{\frac{\pi}{2}} y^2\,dx$, where $y = \sin 2x$. *Ans.* $\frac{\pi}{4}$.

4. $\int_0^{\frac{\pi}{4}} y^2\,dx$, where $y = \tan x$. *Ans.* $1 - \frac{\pi}{4}$.

5. $\int_0^a xy\,dx$, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $\frac{1}{3}a^2b$.

6. $\int_0^1 yx\,dy$, where $y = \log x$. *Ans.* 1.

7. $\int_0^1 yx\,dy$, where $y = \sin x$. *Ans.* $\frac{\pi}{8}$.

8. $\int_0^1 x^2y\,dx$, where $y = e^x$. *Ans.* 0.718.

9. $\int_0^a y\,dx$, where $y = at^2$, $x = at^3$. *Ans.* $\frac{3}{5}a^2$.

10. $\int_0^b yx\,dy$, where $x = a\cos\phi$, $y = b\sin\phi$. *Ans.* $\frac{1}{3}ab^2$.

11. $\int_0^a xy\,dx$, where $x = a\cos\phi$, $y = b\sin\phi$. *Ans.* $\frac{1}{3}a^2b$.

12. $\int_0^a y\,dx$, where $x = a\tan^2\phi$, $y = 2a\tan\phi$. *Ans.* $\frac{4}{3}a^2$.

13. $\int_0^b x^2\,dy$, where $x = a\sec\phi$, $y = b\tan\phi$. *Ans.* $\frac{4}{3}a^2b$.

14. $\int_0^a y^2\,dx$, where $x = a\cos\phi$, $y = b\sin\phi$. *Ans.* $\frac{2}{3}ab^2$.

110. The integral interpreted as an area. Consider the area A bounded by the curve $y = f(x)$, the x-axis, the fixed ordinate $x = a$, and a variable ordinate $x = x$. The function $f(x)$ is here assumed to be one-valued and continuous, and to have only a finite number of maxima and minima in the interval from $x = a$ to $x = b$. In fact, we may assume for definiteness that the curve rises throughout the interval; the argument is readily modified to fit the case when the curve steadily falls, or rises and falls alternately.

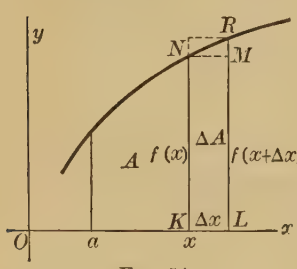


FIG. 54

When x is increased by an amount Δx, A assumes an increment ΔA, the area $KLRN$ in Fig. 54. It appears from the figure that ΔA is numerically greater than the area $f(x)\Delta x$ of the inscribed rectangle $KLMN$, and less than the area $f(x + \Delta x)\Delta x$ of the circumscribed rectangle:

$$f(x)\Delta x < \Delta A < f(x + \Delta x)\Delta x.$$

Hence

$$f(x) < \frac{\Delta A}{\Delta x} < f(x + \Delta x).$$

If now Δx approaches 0, $f(x + \Delta x)$ approaches $f(x)$, and since $\frac{\Delta A}{\Delta x}$ always lies between $f(x)$ and $f(x + \Delta x)$, it must also approach $f(x)$. Thus

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x).$$

Since the derivative of A is $f(x)$, it follows by the definition of the integral that

$$A = \int f(x)dx.$$

In case the position of the fixed ordinate $x = a$ is given, the constant of integration may be determined by the fact that $A = 0$ when $x = a$.

We have thus proved the following result:

The indefinite integral $\int f(x)dx$ *represents the area bounded by the curve* $y = f(x)$, *the x-axis, a fixed ordinate, and a variable ordinate* *.

In particular, the *change* in this area as x changes from a to b is the *definite integral* $\int_a^b f(x)dx$. Hence:

The definite integral $\int_a^b f(x)\,dx$ *represents the area bounded by the curve* $y = f(x)$, *the x-axis, and the lines* $x = a$, $x = b$.

FIG. 55

Example: Find the area bounded by the parabola $y = x^2$, the x-axis, and the lines $x = 1$, $x = 3$.

We have

$$A = \int_1^3 x^2\,dx = \frac{x^3}{3}\Big]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}.$$

* It is evident that if $f(x)$ is continuous, this area always exists; hence *every continuous function has an integral* (cf. § 85).

EXERCISES

In the following, find the area bounded by the x-axis, the given curve, and the indicated ordinates. Check roughly by drawing the figure on coördinate paper and estimating the area.

1. $y = x^3$, $x = 0$, $x = 4$. *Ans.* 64.
2. The parabola $y^2 = 4x$, $x = 0$, $x = 1$. *Ans.* $\frac{4}{3}$.
3. The hyperbola $xy = 1$, $x = 1$, $x = 3$. *Ans.* 1.099.
4. The curve $x^2y = a^3$, $x = a$, $x = 2a$. *Ans.* $\frac{1}{2}a^2$.
5. The curve $y = e^x$, $x = 0$, $x = 1$. *Ans.* 1.718.
6. Find the area between the curve $y = 4 - x^2$ and the x-axis. *Ans.* $\frac{32}{3}$.
7. Find the area bounded by the curve $y = \log x$, the x-axis, and the line $x = 2$. *Ans.* 0.386.
8. Find the area of one arch of the sine curve. *Ans.* 2.

111. Improper integrals. Definite integrals in which either or both of the limits of integration are infinite, and also those in which the integrand becomes infinite within the interval of integration, are called *improper integrals.* Such integrals have no meaning under the definitions so far laid down (see §§ 105, 110); we proceed to show how they may arise, and to find under what conditions a meaning can be assigned to them.

Examples: (*a*) The area under the curve $x^2y = 1$ from $x = 1$ to $x = b$ is evidently

$$A = \int_1^b \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = 1 - \frac{1}{b}.$$

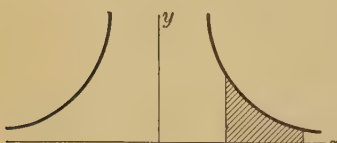


FIG. 56

When b becomes infinite, the area approaches the limit 1. This limit we *define* as the "area bounded by" the curve, the x-axis, and the line $x = 1$, although it is not properly a bounded area in the usual sense of the term. Symbolically we write

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^\infty = 1.$$

(*b*) The area under the curve $xy^2 = 1$ from $x = a\ (a > 0)$ to $x = 1$ is

$$A = \int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

When a approaches 0, the initial ordinate becomes infinite; the area approaches the limit 2. This limit we define as the area in the first quadrant "bounded" by the curve, the axes, and the line $x = 1$. For brevity, we write merely

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_0^1 = 2,$$

but it must be borne in mind that the geometric interpretation is quite different from that of the ordinary integral.

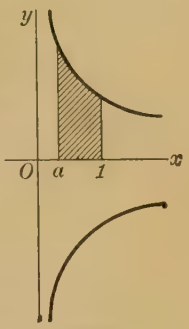


FIG. 57

(*c*) Let us try to find the area under the curve $x^2y = 1$ from $x = -1$ to $x = 1$. If we were to work carelessly, without noticing that the integrand becomes infinite within the interval, we might write

$$A = \int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big]_{-1}^1 = -2,$$

which is obviously absurd. But if we write

$$A = \lim_{a' \to 0} \int_{-1}^{-a'} \frac{dx}{x^2} + \lim_{a'' \to 0} \int_{a''}^1 \frac{dx}{x^2}$$
$$= \lim_{a' \to 0} \left[-\frac{1}{x} \right]_{-1}^{-a'} + \lim_{a'' \to 0} \left[-\frac{1}{x} \right]_{a''}^1,$$

it is clear that the limits do not exist, and the area-integral has no meaning.

These examples suggest the following *definitions:*

1. $$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx;$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx,$$

provided the limits exist.

2. If $f(x)$ becomes infinite as $x \to b^-$,

$$\int_a^b f(x)\,dx = \lim_{b' \to b^-} \int_a^{b'} f(x)\,dx;$$

if $f(x)$ becomes infinite as $x \to a^+$,

$$\int_a^b f(x)\,dx = \lim_{a' \to a^+} \int_{a'}^b f(x)\,dx,$$

provided the limits exist.

3. If $f(x)$ becomes infinite as $x \to c$, where $a < c < b$,

$$\int_a^b f(x)\,dx = \lim_{c' \to c^-} \int_a^{c'} f(x)\,dx + \lim_{c'' \to c^+} \int_{c''}^b f(x)\,dx,$$

provided the limits exist.

It is obvious that an integral with an infinite limit may be interpreted in general as the area under a curve which is asymptotic to the x-axis; an integral whose integrand becomes infinite may usually be thought of as the area between a curve and a vertical asymptote. Of course, as in example (*c*) above, these integrals may not have any meaning in a given case.

EXERCISES

Evaluate the following integrals.

1. $\int_1^\infty \frac{dx}{x^3}$. *Ans.* $\frac{1}{2}$.
2. $\int_{-\infty}^0 e^{2x}\,dx$. *Ans.* $\frac{1}{2}$.
3. $\int_{-\infty}^\infty \frac{du}{1+u^2}$. *Ans.* π.
4. $\int_0^\infty xe^{-\frac{1}{2}x^2}\,dx$. *Ans.* 1.
5. $\int_0^\infty \frac{dx}{x+1}$. *Ans.* No meaning.
6. $\int_{-\infty}^\infty \sin x\,dx$. *Ans.* No meaning.
7. $\int_0^\infty \frac{x\,dx}{x^2+1}$. *Ans.* No meaning.
8. $\int_{\sqrt{2}}^\infty \frac{x\,dx}{x^4+4}$. *Ans.* $\frac{\pi}{16}$.
9. $\int_0^{2a} \frac{dy}{\sqrt{2a-y}}$. *Ans.* $2\sqrt{2a}$.
10. $\int_0^1 \frac{dx}{x}$. *Ans.* No meaning.

11. $\int_{-1}^{1} \frac{dx}{x^3}$. *Ans.* No meaning.

12. $\int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}}$. *Ans.* π.

13. $\int_{-1}^{1} \frac{dx}{x^{\frac{1}{3}}}$. *Ans.* 0.

14. $\int_{0}^{2} \frac{dx}{x - 1}$. *Ans.* No meaning.

15. $\int_{2}^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}$. *Ans.* $\frac{\pi}{6}$.

16. $\int_{1}^{\infty} \frac{dx}{x(x^2 + 1)}$. *Ans.* 0.347.

17. $\int_{1}^{\infty} \frac{dx}{x^2\sqrt{x^2 + 4}}$. *Ans.* $\frac{1}{4}\sqrt{5} - \frac{1}{4}$.

18. $\int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$. *Ans.* $\frac{2}{a^2}$.

19. Trace the curve $y = \frac{a^3}{x^2 + a^2}$, and find the area between the curve and its asymptote. *Ans.* πa^2.

20. Find the area in the second quadrant under the curve $y = e^x$. *Ans.* 1.

CHAPTER XVIII

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

112. Area under a curve. We know by § 110 that the area $PQRS$ (Fig. 58) bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$, $x = b$ is given by the definite integral $\int_a^b f(x)dx$. A new expression for the same area will now be obtained.

We may evidently get an *approximate* expression for the area A by dividing the base PQ into n equal intervals Δx, erecting the ordinates at the points of division, and taking the sum of the inscribed rectangles $PLMS$, etc. The areas of these rectangles are respectively

$$f(x_1)\Delta x,\ f(x_2)\Delta x,\ \cdots,\ f(x_n)\Delta x,$$

where

$$\begin{aligned} x_1 &= a, \\ x_2 &= a + \Delta x, \\ x_3 &= a + 2\,\Delta x, \\ &\vdots \quad \vdots \\ x_n &= a + (n-1)\Delta x = b - \Delta x. \end{aligned}$$

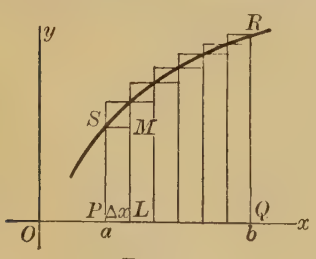


FIG. 58

Hence, *approximately*,

$$A = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Now it is geometrically evident that this sum of rectangles may be made to represent the area A with an error

numerically less than any preassigned constant however small, by taking n sufficiently large. Hence, by the definition of § 11, we have *exactly*

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x. \tag{1}$$

A formal proof of this statement may be given as follows: Let A_1 denote the sum of the inscribed rectangles, A_2 the sum of the circumscribed rectangles (Fig. 59). Then it is obvious that for all values of n

$$A_1 < A < A_2.$$

Now the difference between A_2 and A_1 is the sum of the shaded rectangles. Sliding all these across into the last column, we see that

$$\begin{aligned} A_2 - A_1 &= IR \cdot \Delta x \\ &= [f(b) - f(a)]\,\Delta x. \end{aligned}$$

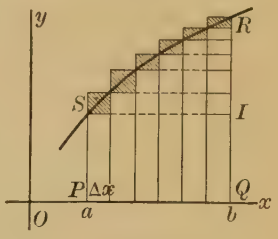


FIG. 59

As n increases indefinitely Δx approaches 0, so that $A_2 - A_1$ also approaches 0. Since A always lies between A_1 and A_2, it follows that A_1 and A_2 approach A as their common limit. Hence we have formula (1).

The n parts into which PQ is divided need not be taken all equal; the same limit is obtained provided the width of each rectangle approaches 0 as the number of divisions is indefinitely increased. Further, it is clear that we may take the limit of the sum of either the inscribed or the circumscribed rectangles, or of any set intermediate between these two.

The rectangles $f(x_i)\Delta x$ are called *elements* of area. As n increases indefinitely, each of the elements approaches 0; *i.e.* they are infinitesimals.

Questions like that of the present article, in which we have to deal with the limit of a sum of infinitesimal elements, will arise many times in this and later chapters. As in every case the existence of the limit will be evident by geometric intuition, we shall in future omit formal proofs.

113. Evaluation of the limit. Equating the values of A found in §§ 110 and 112, we find

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx.$$

Thus the limit occurring in § 112 can always be evaluated by a definite integration.*

The fact that the quantity $f(x)\,dx$ appearing under the integral sign represents the area of a rectangle of altitude $f(x)$ and base $dx = \Delta x$, and thus suggests the sum from which the integral was derived, is the chief reason for using the notation $\int f(x)\,dx$ (see § 84). In fact, the integral sign $\int$ is historically a somewhat conventionalized S, meaning sum.

114. The fundamental theorem. In § 112 we have expressed the area under a plane curve as the limit of a sum of rectangles; in § 110 we have found the same area as a definite integral. It is clear that the arguments used will hold *no matter what may be the geometric or physical meaning of the given function*, for any function whatever may be interpreted as the ordinate of a point on a plane curve. We therefore have at once the following

FUNDAMENTAL THEOREM FOR DEFINITE INTEGRALS:

Given a function $f(x)$, continuous in the interval from $x = a$ to $x = b$, divide this interval into n equal parts Δx, and form the sum $\sum_{i=1}^{n} f(x_i)\,\Delta x$, where $x_1 = a$, $x_2 = a + \Delta x$, $\cdots$, $x_n = a + (n-1)\Delta x$. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx.$$

* More precisely, the limit can always be *expressed* as a definite integral; the actual evaluation of the integral is often impossible.

*Hence, if in any problem an arbitrarily close approximation * to the required quantity can be found by adding up terms of the type $f(x)\,\Delta x$ from $x = a$ to $x = b$, that quantity is given exactly by the definite integral* $\int_a^b f(x)\,dx$.

We have now presented the definite integral in two distinct aspects: first, as the change in the value of the indefinite integral between two values of the variable; second, as the *limit of a sum of infinitesimal elements*. The great advantage of this latter point of view will become apparent as we proceed.

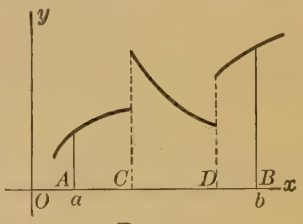


FIG. 60

It may be remarked that if the function $f(x)$ has a finite number of finite discontinuities in the interval from a to b, as in Fig. 60, it can still be integrated. For it will be continuous in a number of subintervals such as AC, CD, DB, to each of which the fundamental theorem can be applied and the results added.

115. Plane areas in cartesian coördinates. Not only the area considered in § 112, but any plane area bounded by curves whose equations are given in cartesian coördinates, can be found as follows. Imagine inscribed in the area a set of n elementary rectangles of altitude h_i and width Δl in such a way that the sum $\sum_{i=1}^{n} h_i\,\Delta l$ may be made to represent the area to any desired degree of approximation by increasing n. Then at once, by the fundamental theorem,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} h_i \Delta l = \int h\,dl,$$

* That is, an approximation in which the error may be made numerically less than any preassigned positive constant.

the limits being chosen in such a way as to extend the integration over the whole area. Of course in any particular problem h and dl must be expressed in terms of the coördinates.

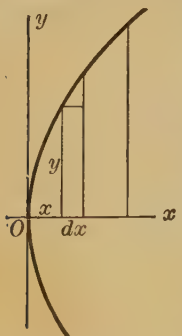


FIG. 61

In every problem the student should make a sketch of the area to be found, draw an element in a general position, and obtain the area of the element directly from the figure, as in the following

Examples: (*a*) Find the area in the first quadrant bounded by the parabola $y^2 = 4ax$, the x-axis, and the line $x = a$.

An arbitrarily close approximation to this area can be found by forming a sum of rectangles of altitude y, base dx, and area $y\,dx$, as shown in Fig. 61. Hence, we have *exactly*

$$A = \int_0^a y\,dx = 2\int_0^a \sqrt{ax}\,dx = \frac{2}{a}\cdot\frac{2}{3}(ax)^{\frac{3}{2}}\Big]_0^a = \frac{4}{3}a^2.$$

(*b*) The above example can also be easily solved by taking the element parallel to Ox (*i.e.* with its finite side parallel to Ox), as in Fig. 62. The base of the rectangle is evidently $a - x$, the altitude dy, whence

$$A = \int_0^{2a} (a - x)\,dy = \int_0^{2a}\left(a - \frac{y^2}{4a}\right)dy$$

$$= ay - \frac{y^3}{12a}\Big]_0^{2a} = \frac{4}{3}a^2.$$

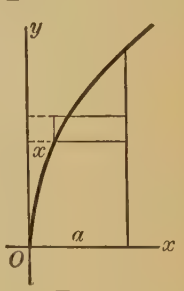


FIG. 62

EXERCISES

1. Find the area bounded by the curve $y = x^3$, the x-axis, and the line $x = 4$. Solve in two ways. *Ans.* 64.

2. Find the area bounded by the curve $y = x^4$, the x-axis, and the line $x = 1$. Solve in two ways. *Ans.* $\frac{1}{5}$.

3. Find the area bounded by the parabola $y = x^2$, the y-axis, and the lines $y = 1$, $y = 3$. *Ans.* 2.80.

4. Find the area bounded by the curve $ay^2 = x^3$ and the line $x = a$. *Ans.* $\frac{4}{5}a^2$.

5. Find the area of half an arch of the curve $y = \cos x$. *Ans.* 1.

6. Find the area of one arch of the curve $y = \frac{1}{3}\cos 2x$. *Ans.* $\frac{1}{3}$.

7. Find the area of one arch of the curve $y = a \sin kt$. *Ans.* $\frac{2a}{k}$.

8. Find the area bounded by the curves $2y = x^2$, $x - y + 4 = 0$. *Ans.* 18.

9. Find the area bounded by the curve $y = (x - 4)^2(x - 3)$ and the x-axis. *Ans.* $\frac{1}{12}$.

10. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. *Ans.* $\frac{16}{3}a^2$.

11. Show that the area bounded by a parabola and any chord perpendicular to the axis is two thirds of the circumscribing rectangle.

12. Find the area under the catenary $y = a \cosh \frac{x}{a}$ from $x = -a$ to $x = a$. *Ans.* $2a^2 \sinh 1$.

13. Find the area between the curve $x^2 = 4a^2 - ay$ and the x-axis, taking the element (a) parallel to Oy; (b) parallel to Ox. *Ans.* $\frac{32}{3}a^2$.

14. Find the area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. *Ans.* $\frac{1}{6}a^2$.

15. Find the area bounded by the curve $x = at^2$, $y = at^3$, the x-axis, and the line $x = 4a$. *Ans.* $\frac{64}{5}a^2$.

16. Find the area of a triangle by integration.

17. Find the area of a circle.

18. Find the area of an ellipse. *Ans.* πab.

19. Solve Ex. 18, using the parametric equations $x = a \cos \phi$, $y = b \sin \phi$.

20. Find the area in the first quadrant under the curve $y = xe^{-\frac{1}{2}x^2}$. *Ans.* 1.

21. Find the area bounded by the curve $y = xe^x$ and its asymptote. *Ans.* 1.

22. Find in two ways the area bounded by the curve $y = (x^2 - 1)^2$ and the x-axis. *Ans.* $\frac{16}{15}$.

23. Find the area bounded by the hyperbola $x = a \sec \phi$, $y = a \cos \phi$, the x-axis, and the lines $x = a$, $x = 2a$. *Ans.* $a^2 \log 2$.

24. Find the area between the parabolas $y^2 = 2ax$, $y^2 = 4ax - a^2$. *Ans.* $\frac{1}{3}a^2$.

25. Find the area bounded by the curve $y = \log x$ and the coördinate axes. Solve in various ways. *Ans.* 1.

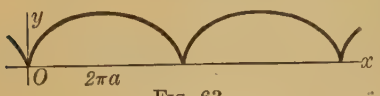


FIG. 63

26. Find the area of one arch of the cycloid (Fig. 63)

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Ans. $3\pi a^2$.

27. Find the area bounded by the curve $y = \dfrac{2x}{1 + x^2}$, its asymptote, and the maximum ordinate. *Ans.* 0.693.

28. Trace the curve $y^2(x^2 + a^2) = a^2x^2$, and find the area bounded by the curve and the line $x = a$. *Ans.* $0.83\,a^2$.

29. Trace the curve $ay^2 = ax^2 + x^3$, and find the area of the loop. *Ans.* $\frac{8}{15}\,a^2$.

30. Trace the curve $y = \dfrac{3 - 2x}{4 + x^2}$, and find the area under the curve from $x = -2$ to $x = 0$. *Ans.* 1.87.

31. Find the area bounded by the curve $y = \dfrac{\log x}{x}$, the x-axis, and the maximum ordinate. *Ans.* $\frac{1}{2}$.

32. Find the area in the first quadrant under the curve $y^2 = \dfrac{x^2}{x - 1}$, between the minimum ordinate and the ordinate $x = 3$. *Ans.* 2.05.

33. Trace the curve $y^2 = x^4(x + 4)$, and find the area inclosed by it. *Ans.* $\dfrac{2^{12}}{105}$.

34. Find the area in the first quadrant under the curve $y = \dfrac{4x}{x^4 + 3}$. Show that the maximum ordinate trisects this area. *Ans.* $\frac{1}{3}\sqrt{3}\,\pi$.

35. Find the area between the curve

$y^3 = x^3(1 - x)$ and Ox. *Ans.* $\frac{9}{28}$.

36. Find the area of a circular segment of height h.

Ans. $\frac{1}{2}\pi a^2 - (a - h)\sqrt{2ah - h^2} - a^2 \arcsin\dfrac{a - h}{a}$.

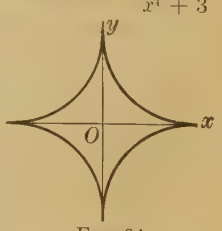


FIG. 64

37. Find the area inside the four-cusped hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$ (Fig. 64). *Ans.* $\frac{3}{8}\pi a^2$.

38. Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$. *Ans.* $[2\sqrt{3} - \log(2 + \sqrt{3})]a^2$.

39. Solve Ex. 38, using the parametric equations $x = a\sec\phi$, $y = a\tan\phi$.

40. In Ex. 18, if the distance between the directrices is given, when is the area of the curve a maximum? *Ans.* $b = \frac{1}{3}\sqrt{3}\,a$.

116. Plane areas in polar coördinates. Given the equation

$$r = f(\theta)$$

of a plane curve in polar coördinates, let us try to find the area bounded by the curve and the radii vectores corresponding to $\theta = \alpha$, $\theta = \beta$. We can obtain an arbitrarily close approximation to the area by inscribing in it n circular sectors of radius r_i and angle $\Delta\theta$, hence

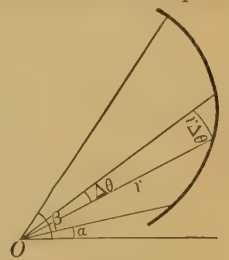


FIG. 65

of area * $\frac{1}{2} r_i^2 \Delta\theta$, and forming the sum $\sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta$. Hence, by the fundamental theorem,

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \tfrac{1}{2} r_i^2 \Delta\theta = \tfrac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta.$$

By means of a theorem to be proved in § 119, it can be shown that this result may also be obtained by choosing as the element a *triangle* of altitude r_i, base $r_i \Delta\theta$, and area $\frac{1}{2} r_i^2 \Delta\theta$.

EXERCISES

1. Find the area swept out by the radius vector of the spiral of Archimedes $r = a\theta$, in the interval from $\theta = 0$ to $\theta = 2\pi$. *Ans.* $\frac{4}{3}\pi^3 a^2$.

2. Solve Ex. 1 for the logarithmic spiral $\log r = k\theta$. *Ans.* $\frac{1}{4k}(e^{4\pi k} - 1)$.

3. Find the area of the curve $r^2 = a^2 \cos\theta$. *Ans.* $2a^2$.

4. Find the area inside the curve $r^2 = a^2 \cos 2\theta$. *Ans.* a^2.

5. Find the area of the cardioid $r = a(1 - \cos\theta)$. *Ans.* $\frac{3}{2}\pi a^2$.

6. Find the area of the circle $r = 2a \cos\theta$. *Ans.* πa^2.

7. Find the area between the parabola $r = \dfrac{2a}{1 + \cos\theta}$ and its latus rectum. (Note that $1 + \cos\theta = 2\cos^2 \frac{1}{2}\theta$.) *Ans.* $\frac{8}{3} a^2$.

8. Find the area of the curve $r = a \cos 2\theta$. *Ans.* $\frac{1}{2}\pi a^2$.

* By elementary geometry, the area of a circular sector of radius r and angle α is

$$A = \tfrac{1}{2} r^2 \alpha.$$

9. Find the area inside the curve $r \sin \frac{1}{3}\theta = a$ below the initial line. *Ans.* $\sqrt{3}\, a^2$.

10. Find the area of the curve $r = a \sin^2 \theta$. *Ans.* $\frac{3}{8}\pi a^2$.

11. Find the area of the curve $r = a \cos 3\theta$. *Ans.* $\frac{1}{4}\pi a^2$.

12. Find the area bounded by the curve $r \cos 2\theta = a$ and the lines $\theta = 0, \theta = \frac{\pi}{3}$. *Ans.* $\frac{1}{4}\sqrt{3}\, a^2$.

Solve the following by transforming the equations of the bounding curves to polar coördinates.

13. Find the area in the first quadrant bounded by the curves $y = x, y = x^3$. *Ans.* $\frac{1}{4}$.

14. Find the area bounded by the curves $y = x, \quad y = 2x, \quad y = x^2$. *Ans.* $\frac{7}{6}$.

15. Find the area in the first quadrant bounded by the curve $y^3 = x^3(1 - x^2)$ and the x-axis. *Ans.* $\frac{3}{8}$.

16. Find the area bounded by the curve $y^3 = x^3(1 - x)$ and the x-axis. *Ans.* $\frac{9}{28}$.

17. Find the area of the loop of the curve $ay^2 = ax^2 + x^3$. *Ans.* $\frac{8}{15}\, a^2$.

117. Solids of revolution. The volume of a solid of revolution may be found very readily by the fundamental theorem.

Suppose the volume is generated by revolving the area

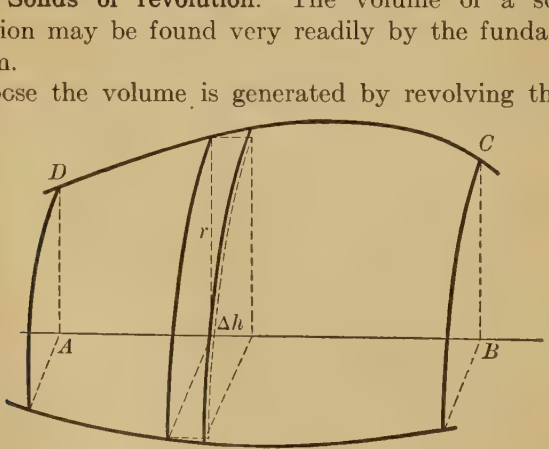


FIG. 66

$ABCD$ about the line AB. If we inscribe in the revolving area a set of n rectangles of altitude r_i and base Δh, each rectangle will generate in its rotation a *circular disk*, or cylinder, of radius r_i, altitude Δh, and volume $\pi r_i^2 \Delta h$. Further, as n

increases the sum of these cylindrical elements approaches as its limit the required volume. Hence

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi r_i^2 \Delta h = \pi \int r^2 dh,$$

the limits being chosen so as to include the whole volume. Of course in any problem both r and dh must be expressed in terms of the coördinates.

When the axis of revolution does not form part of the boundary of the revolving area, we may choose as elements a set of *circular rings*, as in example (*b*) below.

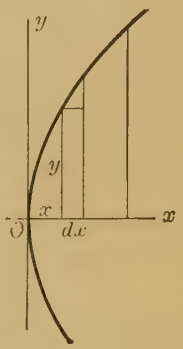


Fig. 67

Examples: (*a*) The area bounded by a parabola, its axis, and its latus rectum revolves about the axis. Find the volume generated.

Dividing the area into elements as in Fig. 67, we see that each rectangle generates a cylindrical volume-element of radius y, altitude dx, and volume $\pi y^2 dx$. Hence

$$V = \pi \int_0^a y^2 \, dx = 4\pi a \int_0^a x \, dx = 2\pi a^3.$$

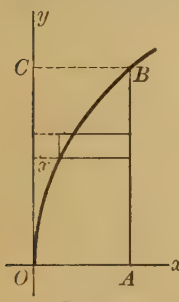


Fig. 68

(*b*) The above area rotates about Oy. Find the volume generated.

If we divide the area into elements as in Fig. 68, each element generates a *circular ring* of outer radius a, inner radius x, thickness dy, and volume $\pi(a^2 - x^2)dy$. Further, the limit of the sum of these volumes is the required volume. Hence

$$V = \pi \int_0^{2a} (a^2 - x^2) dy = \pi \int_0^{2a} \left(a^2 - \frac{y^4}{16\, a^2} \right) dy$$

$$= \pi \left[a^2 y - \frac{y^5}{80\, a^2} \right]_0^{2a} = \tfrac{8}{5}\, \pi a^3.$$

This result could have been obtained equally well by finding the volume generated by rotating the area OBC about Oy, and subtracting this from the volume of the cylinder formed by revolving the rectangle $OABC$. But in case it is possible to simplify the integral before integrating, as often happens, the first method is to be preferred.

118. Solids of revolution: second method. The following method for finding volumes of solids of revolution is often preferable to that of § 117.

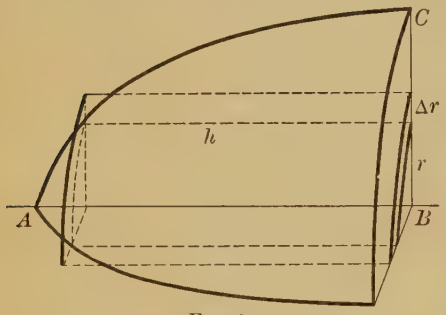


FIG. 69

Let us take as an element of the revolving area ABC (Fig. 69) a rectangle *parallel* to the axis of revolution AB, of length h_i and width Δr. This rectangle generates by its rotation a *cylindrical shell* of inner radius r_i, altitude h_i, and thickness Δr. The volume of the shell is evidently

$$\pi(r_i + \Delta r)^2 h_i - \pi r_i^2 h_i = 2\,\pi r_i h_i \Delta r + \pi h_i \overline{\Delta r}^2,$$

and the limit of the sum of these elementary shells is evidently the required volume. Now it will be shown in the next article that, in passing to the limit, we may *neglect the infinitesimal of higher order* * $\pi h_i \overline{\Delta r}^2$. Hence, by the fundamental theorem,

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\,\pi r_i h_i \Delta r = 2\,\pi \int rh\,dr,$$

the integration being extended through the whole region.

* This is easily shown directly. The quantity neglected is $\sum_{i=1}^{n} \pi h_i \Delta r^2$. This may be written $\pi \Delta r \sum_{i=1}^{n} h_i \Delta r$. When Δr approaches 0, the sum $\sum_{i=1}^{n} h_i \Delta r$ approaches a finite limit, viz. the generating area, so that the whole quantity approaches 0.

This result is easily remembered from the fact that the integrand is the differential of volume of a right circular cylinder, the altitude being constant:

$$V = \pi r^2 h,\ dV = 2\pi rh\, dr.$$

That is, *the volume * of a thin cylindrical shell is equal to the circumference times the height times the thickness.*

Example: Solve example (*b*), § 117, by the present method.

Divide the rotating area into rectangles parallel to Oy, as in Fig. 67. Each rectangle generates a cylindrical shell of radius x, altitude y, and thickness dx. Hence

$$V = 2\pi \int_0^a xy\, dx = 4\pi a^{\frac{1}{2}} \int_0^a x^{\frac{3}{2}}\, dx = \tfrac{8}{5}\pi a^3.$$

119. A theorem on infinitesimals. It often happens, as in the preceding article, that in applying the fundamental theorem we have to replace the element as originally chosen by another element differing from the first one by an infinitesimal of higher order. That this is allowable appears from the following

THEOREM: *The limit of a sum of positive infinitesimals is unchanged when each infinitesimal is replaced by another that differs from it by an infinitesimal of higher order.*

It follows that, in taking the limit of such a sum, *all infinitesimals of higher order may be neglected,* as was done in § 118.

In this connection it should perhaps be mentioned explicitly that two infinitesimals differ from each other by an infinitesimal of higher order whenever the limit of their ratio is 1, and conversely.

To prove the theorem, let $u_1, u_2, \cdots, u_n$ be a set of positive infinitesimals such that $\lim_{n\to\infty} \sum_{i=1}^{n} u_i$ exists; and let

* Apart from infinitesimals of higher order. Cf. Ex. 1, p. 93.

$v_1, v_2, \cdots, v_n$ be another set such that v_i differs from u_i by an infinitesimal of higher order:

$$v_i = u_i + w_i u_i,$$

where w_i is infinitesimal. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_i = \lim_{n\to\infty} \sum_{i=1}^{n} u_i + \lim_{n\to\infty} \sum_{i=1}^{n} w_i u_i.$$

Denoting by $\overline{w}$ the absolute value of the largest of the w's, we have

$$-\overline{w} \sum_{i=1}^{n} u_i \leqq \sum_{i=1}^{n} w_i u_i \leqq \overline{w} \sum_{i=1}^{n} u_i.$$

Since the first and third of these quantities both approach 0, the second must do likewise. Hence

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_i = \lim_{n\to\infty} \sum_{i=1}^{n} u_i,$$

and the theorem is proved.

EXERCISES

1. Find the volume of a sphere. Solve in two ways.

2. The area bounded by the hyperbola $x^2 - y^2 = a^2$, the x-axis, and the line $x = 2a$ revolves about Ox. Find the volume generated.

Ans. $\frac{4}{3}\pi a^3$.

3. Find the volume generated by revolving about Ox the area in the second quadrant under the curve $y = e^x$. *Ans.* $\frac{1}{2}\pi$.

4. By two methods, find the volume of an oblate spheroid.

Ans. $\frac{4}{3}\pi a^2 b$.

5. Find the volume of a prolate spheroid, using the equations $x = a\cos\phi$, $y = b\sin\phi$. Solve in two ways. *Ans.* $\frac{4}{3}\pi ab^2$.

6. Find the volume generated by revolving about Ox the area bounded by the hyperbola $xy = a^2$, the line $x = a$, and the x-axis.

Ans. πa^3.

7. Solve Ex. 6, using for the hyperbola the equations $x = a\cot\phi$, $y = a\tan\phi$.

8. Solve Ex. 6, using the equations $x = a\sec\phi$, $y = a\cos\phi$.

9. The area bounded by the curve $y = \log x$, the axes, and the line $y = -1$ revolves about Oy. Find the volume generated.

Ans. $\frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$.

10. Find the volume of a right circular cone. Solve in two ways.

11. Find the volume generated by revolving about Oy the area in the second quadrant under the curve $y = e^x$. Solve in two ways.

Ans. 2π.

12. The area bounded by a parabola and its latus rectum revolves about the latus rectum. Find the volume generated, by two methods.

Ans. $\frac{32}{15}\pi a^3$.

13. A hyperboloid is generated by revolving the hyperbola $x^2 - y^2 = a^2$ about Oy. Find the volume outside the hyperboloid, and inside a cylinder of radius $2a$ whose axis is the y-axis. *Ans.* $4\pi\sqrt{3}\,a^3$.

14. Solve Ex. 13, using the equations $x = a \sec \phi$, $y = a \tan \phi$.

15. A hyperboloid is generated by revolving the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ about Oy. Find the volume included between the planes $y = 0$, $y = b$.

Ans. $\frac{4}{3}\pi a^2 b$.

16. Solve Ex. 15, using the equations $x = a \sec \phi$, $y = b \tan \phi$.

17. Find the volume generated by revolving about the y-axis the area bounded by the parabola $y^2 = 4 - x$ and the y-axis. Solve in two ways. *Ans.* $\frac{512}{15}\pi$.

18. The area under one arch of the sine-curve revolves about the x-axis. Find the volume generated. *Ans.* $\frac{1}{2}\pi^2$.

19. The area under one arch of the sine curve revolves about the y-axis. Find the volume generated. *Ans.* $2\pi^2$.

20. Find the volume generated by revolving the area under the curve $y = e^x$, from $x = 0$ to $x = 1$, about the line $x = 1$. *Ans.* $2\pi(e - 2)$.

21. Find the volume obtained by revolving about Oy the area under the catenary $y = a \cosh \frac{x}{a}$, from $x = 0$ to $x = a$. *Ans.* $2\pi a^3\left(1 - \frac{1}{e}\right)$.

22. Trace the curve $y^2 = 2x(x - 1)(x - 2)$, and find the closed volume generated by revolving the curve about the x-axis. *Ans.* $\frac{1}{2}\pi$.

23. Trace the curve $a^2y^2 = x^3(2a - x)$, and find the volume formed by revolving the curve about the x-axis. *Ans.* $\frac{8}{5}\pi a^3$.

24. The area OBC of Fig. 68 revolves about the line CB. Find the volume obtained, by two methods. *Ans.* $\frac{2}{3}\pi a^3$.

25. The area bounded by the curve $y = x^3 - 8$ and the coördinate axes revolves about the y-axis. Find the volume generated, by two methods. *Ans.* $\frac{96}{5}\pi$.

26. Find the volume generated by revolving about the line $x = 2a$ the area bounded by that line, the x-axis, and the curve $a^2y = x^3$. Solve in two ways. *Ans.* $\frac{128}{15}\pi a^3$.

27. By two methods, find the volume formed by revolving about Oy the area between the curve $y = e^{-\frac{1}{2}x^2}$ and its asymptote. *Ans.* 2π.

28. Find the volume obtained by revolving about Ox the area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. Solve in two ways. *Ans.* $\frac{1}{105}\pi a^3$.

29. Solve Ex. 28, using the equations $x = a\cos^4\phi$, $y = a\sin^4\phi$.

30. Find the volume formed by revolving the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Fig. 64, p. 182) about Ox. *Ans.* $\frac{32}{105}\pi a^3$.

31. Solve Ex. 30 by two methods, using the equations $x = a\cos^3\phi$, $y = a\sin^3\phi$.

32. The area bounded by the curve $y = (4 - x^2)^2$ and the x-axis revolves about the y-axis. Find the volume generated, by two methods. *Ans.* $\frac{64}{3}\pi$.

33. A round hole of radius a is bored through the center of a sphere of radius $2a$. Find the volume cut out. *Ans.* $\frac{4}{3}\pi(8 - 3\sqrt{3})a^3$.

34. Find the volume generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ (Fig. 63, p. 182) about the x-axis. *Ans.* $5\pi^2a^3$.

35. Find the volume of the torus formed by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$. *Ans.* $2\pi^2a^2b$.

36. Solve Ex. 35, using the equations $x = a\sin\theta$, $y = a\cos\theta$.

37. Find the volume generated by revolving an ellipse about the tangent at the end of the minor axis. *Ans.* $2\pi^2ab^2$.

38. Solve Ex. 37, using the equations $x = a\cos\phi$, $y = b\sin\phi$.

39. Trace the curve $a^4y^2 = a^2x^4 - x^6$, and find the volume generated by revolving one loop about Oy. *Ans.* $\frac{8}{105}\pi a^3$.

40. Find the closed volume generated by revolving the curve $y^2 = x^4(x + 4)$ about the y-axis. *Ans.* $\frac{2^{16}}{315}\pi$.

41. Find in two ways the volume generated by revolving about the y-axis the area bounded by the curve $y = \frac{\sin x}{x}$ and the coördinate axes. *Ans.* 4π.

42. Find the volume of a spherical segment of height h.
Ans. $\frac{1}{3}\pi h^2(3a - h)$.

43. Water is poured into a hemispherical bowl of radius 4 in., at the rate of 2 cu. in. per sec. Using the result of Ex. 42, find the rate at which the surface is rising, when the water is 2 in. deep.
Ans. $\frac{1}{6\pi}$ in. per sec.

44. Find the volume cut off from the paraboloid $x^2 + y^2 = 4az$ by the plane $z = c$. *Ans.* $2\pi ac^2$.

45. Find the volume inside the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1$, between the planes $z = 0$, $z = c$. *Ans.* $\frac{4}{3}\pi a^2 c$.

46. Find the volume cut off from the hyperboloid $x^2 - y^2 - z^2 = a^2$ by the plane $x = 3a$. *Ans.* $\frac{20}{3}\pi a^3$.

47. Find the volume inside the cylinder $x^2 + y^2 = 2a^2$ and outside the hyperboloid $x^2 + y^2 - z^2 = a^2$. *Ans.* $\frac{4}{3}\pi a^3$.

48. Find the volume bounded by the surfaces $x^2 + y^2 = 4az$, $x^2 + y^2 = z^2$. *Ans.* $\frac{32}{3}\pi a^3$.

49. Find the volume bounded by the cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z^2}{c^2}$ and the cylinder $x^2 + y^2 = a^2$. *Ans.* $\frac{4}{3}\pi a^2 c$.

50. Find the volume bounded by the surfaces $x^2 + y^2 = az$, $x^2 + y^2 = a^2$, $z = 0$. *Ans.* $\frac{1}{2}\pi a^3$.

51. Find the volume common to the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = a^2$. *Ans.* $\frac{4}{3}\pi a^3(1 - \frac{1}{2}\sqrt{2})$.

120. Miscellaneous solids. The volume of any solid can be expressed as a definite integral, provided we know the area of every plane section parallel to some fixed plane. Let us divide the volume into slices of thickness Δh by means of n planes parallel to this fixed plane. If on each of these plane sections we erect a cylinder of altitude Δh and base A_i, where A_i is the area of the section, the sum of the n cylindrical volumes thus formed will be approximately the required volume, and the limit of this sum will be exactly the volume. Hence

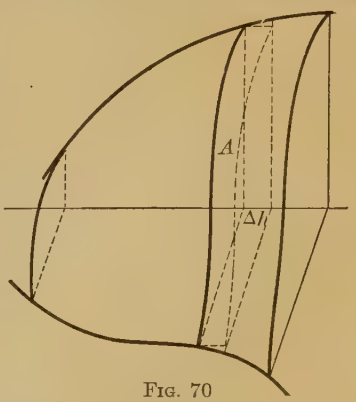


FIG. 70

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A_i \, \Delta h = \int A \, dh,$$

with properly chosen limits.

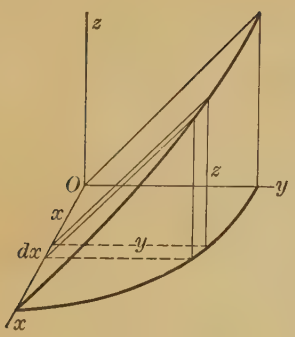


FIG. 71

Example: A woodsman chops halfway through a tree 4 ft. in diameter, one face of the cut being horizontal, the other inclined at 45°. Find the volume of wood cut out.

The figure shows one half of the required volume. If we slice up the volume by planes parallel to the yz-plane, the element is a triangular plate of width y, altitude z, and thickness dx. Hence

$$V = 2\int_0^2 \tfrac{1}{2}\, yz\, dx.$$

But

$$z = y, \text{ and } y = \sqrt{4 - x^2},$$

so that

$$V = \int_0^2 (4 - x^2)\, dx = 5\tfrac{1}{3} \text{ cu. ft.}$$

EXERCISES

1. Solve the above example in a different way.

2. Find the volume of a tetrahedron with three mutually perpendicular faces. *Ans.* $\frac{1}{6}\, abc$.

3. Find the volume sliced off from a right circular cylinder by a plane through a diameter of one base and tangent to the other base. Solve in two ways. *Ans.* $\frac{2}{3}\, a^2h$.

4. Find the volume of a right pyramid with a square base.

5. Solve Ex. 4 by another method.

6. Find the volume of an ellipsoid, using the answer to Ex. 18, p. 181. *Ans.* $\frac{4}{3}\, \pi abc$.

7. Find the volume of an elliptic cone. *Ans.* $\frac{1}{3}\, \pi abh$.

8. Find the volume cut from the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{4z}{c}$ by the plane $z = c$. *Ans.* $2\, \pi abc$.

9. Find the volume inclosed by the hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the planes $y = -b$, $y = b$. *Ans.* $\frac{8}{3}\, \pi abc$.

10. Find the volume of a spherical wedge. (See the footnote, p. 183.) *Ans.* $\frac{2}{3}\alpha a^3$.

11. Find the volume of a wedge cut from a right circular cone by two planes through the axis. *Ans.* $\frac{1}{6}\alpha a^2 h$.

12. By two methods, find the volume common to two equal cylinders of revolution whose axes intersect at right angles. *Ans.* $\frac{16}{3}a^3$.

13. A carpenter chisels a square hole of side 2 in. through a round post of radius 2 in., the axis of the hole intersecting that of the post at right angles. Find the volume of wood cut out. *Ans.* 15.3 cu. in.

14. Find the volume cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = x$, $z = 2x$. Solve in two ways.

15. Find the volume in the first octant under the plane $z = y$ and inside the cylinder $y^2 = 4 - x$. Solve in two ways. *Ans.* 4.

16. Find the volume cut from the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ by the plane $x = 3a$. *Ans.* $\frac{20}{3}\pi abc$.

17. Find the volume in the first octant bounded by the surfaces $x = 1$, $x^2 = y + 2z$. *Ans.* $\frac{1}{20}$.

18. Find the volume in the first octant inside the cylinder $x^2 + z^2 = 1$ and outside the cylinder $y = x^2$. *Ans.* $\frac{1}{16}\pi$.

19. A surface is generated by a line parallel to the zx-plane intersecting the parabola $y^2 = 4ax$, $z = 0$ and the line $z = b$, $x = 0$. Find the volume in the first octant bounded by this surface and the plane $y = a$. *Ans.* $\frac{1}{24}a^2b$.

20. Find the volume in the first octant bounded by the hyperbolic paraboloid generated by a straight line moving always parallel to the xy-plane and passing through the lines $y + z = a$ in the yz-plane and $x = b$ in the xz-plane. *Ans.* $\frac{1}{4}a^2b$.

21. In Ex. 20, derive the equation of the paraboloid; then find the volume by a second method. *Ans.* $xz - ax - by - bz + ab = 0$.

22. Solve Ex. 20 if the line $x = b$ is replaced by the line $x = z$. *Ans.* $\frac{1}{12}a^3$.

23. A hyperbolic paraboloid is generated by a line moving parallel to the zx-plane intersecting the lines $x = a$, $z = 0$ and $z = y$, $x = 0$. Find the volume inclosed by this surface, the coördinate planes, and the plane $y = b$. *Ans.* $\frac{1}{4}ab^2$.

24. A hyperbolic paraboloid is generated by a line moving parallel to the zx-plane and intersecting the lines $x = y = z$ and $x = a$, $z = 0$. Find the volume bounded by this surface, the xy-plane, and the plane $y = x$. *Ans.* $\frac{1}{12}a^3$.

25. A right circular conoid is generated by a straight line which moves always parallel to the xy-plane and passes through the line $y = h$ in the yz-plane and the circle $x^2 + z^2 = a^2$ in the xz-plane. Find the volume of the conoid. *Ans.* $\frac{1}{2}\pi a^2 h$.

26. In Ex. 25, derive the equation of the conoid; then find the volume by a second method. *Ans.* $h^2x^2 = (a^2 - z^2)(h - y)^2$.

27. In Ex. 25, find the shape of the conoid of maximum volume, if the length of the longest generator is given. *Ans.* $a = \sqrt{2}\, h$.

28. Solve Ex. 25 if the line $y = h$ is replaced by the line $y + z = h$ $(h > a)$. *Ans.* $\frac{1}{2}\pi a^2 h$.

29. A surface is generated by a line parallel to the xy-plane intersecting the parabolas $z^2 = x$, $y = 0$ and $z^2 = 2y$, $x = 0$. Find the volume in the first octant bounded by this surface and the plane $z = 1$. *Ans.* $\frac{1}{20}$.

30. Find the volume cut out of the first octant by the cylinders $z = 1 - x^2$, $x = 1 - y^2$. *Ans.* $\frac{13}{35}$.

31. Find the volume in the first octant under the surface $z = xy$ bounded by the cylinder $y = x^2$ and the plane $y = 1$. *Ans.* $\frac{1}{8}$.

32. Find the volume in the first octant bounded by the cylinders $ay = x^2$, $x^2 + az = a^2$. *Ans.* $\frac{2}{15}a^3$.

33. A cylinder is generated by a line moving always parallel to the line $y + z = a$, $x = 0$ and following the circle $x^2 + y^2 = a^2$, $z = 0$. Find the volume in the first octant inside the cylinder. *Ans.* $\frac{1}{3}a^3$.

34. A cylinder is generated by a line moving always parallel to the line $x + 2y = 2a$, $z = 0$ and following the ellipse $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$, $x = 0$. Find the volume in the first octant inside the cylinder. *Ans.* $\frac{2}{3}a^2 b$.

35. A cylinder is generated by a line moving always parallel to the line $x + z = a$, $y = 0$, and following the curve $y^2 + az = a^2$, $x = 0$. Find the volume in the first octant inside the cylinder.

36. The vertex of a cone is at $(a, 0, 0)$; the base is the curve $y^2 + z^2 = 2by$, $x = 0$. Find the volume of the cone. *Ans.* $\frac{1}{3}\pi ab^2$.

37. The vertex of a cone is at $(0, 0, 0)$; the base is the curve $4x^2 + z^2 = 2az$, $y = a$. Find the volume of the cone. *Ans.* $\frac{1}{6}\pi a^3$.

38. A round hole of radius 1 in. is bored centrally through a square timber of side 4 in., the axis of the hole intersecting the edge of the timber at right angles. Find the volume cut out. *Ans.* 15.1 cu. in.

39. The base of a right prism is an isosceles right triangle. By two methods, find the volume cut from the prism by two planes through a side of the base making angles of 30° and 60°, respectively, with the base. *Ans.* $\frac{2}{3}\sqrt{3}\, a^3$.

121. Length of a curvilinear arc. To find the length of the arc of a plane curve C between two given points, or, as the process is often called, to *rectify* the curve, let us inscribe in C a broken line of n segments $\Delta s_i'$ having equal projections * on the x-axis, and form the sum $\sum_{i=1}^{n} \Delta s_i'$. This sum is of course the length of the broken line, and its limit is the length s of the arc. This limit is denoted by the symbol $\int_C ds$: *i.e.*

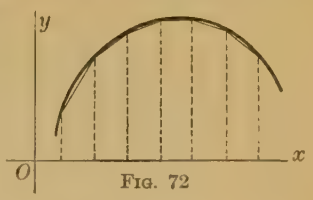


FIG. 72

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i' = \int_C ds.$$

To evaluate this limit, we replace ds by its value in terms of x, y or some other suitable variable, and then integrate between limits in such a way as to extend the integration over the given arc.

Thus, if x is chosen as the variable of integration, we replace ds by the value (§ 62)

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

and obtain

$$\int_C ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

where a and b are the abscissas of the end points of C†, and where $\frac{dy}{dx}$ must be replaced by its value in terms of x from

* It is of course merely for convenience that the broken-line segments are drawn with equal projections on Ox. The division may be made in any manner provided as n increases every segment approaches 0.

† It is assumed that no parallel to the y-axis can meet C in more than one point. If this condition is not satisfied, C must consist of several portions for each of which the condition holds, and each portion may be considered separately.

the equation of the curve*.

To integrate with respect to y, we put

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy,$$

and express the integrand in terms of y.

If x and y are given in terms of a parameter t, we use

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXERCISES

1. Rectify the curve $ay^2 = x^3$ from $x = 0$ to $x = 5a$. *Ans.* $\frac{335}{27}a$.
2. Rectify the catenary $y = a \cosh \frac{x}{a}$ from $x = 0$ to $x = x$. *Ans.* $a \sinh \frac{x}{a}$.
3. Find the length of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $6a$.
4. Solve Ex. 3, using the equations $x = a \sin^3 \phi$, $y = a \cos^3 \phi$.
5. Solve Ex. 1, using the equations $x = at^2$, $y = at^3$.
6. Trace the curve $9y^2 = 4(1 + x^2)^3$, and find its length from $x = 0$ to $x = 2$. *Ans.* $\frac{22}{3}$.

* This step may be justified as follows. We have

$$\Delta s_i' = \sqrt{\Delta x^2 + \Delta y_i^2} = \sqrt{1 + \frac{\Delta y_i^2}{\Delta x^2}}\, \Delta x.$$

Now the limit of the sum $\sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x^2}}\, \Delta x$ cannot be expressed directly as a definite integral by the theorem of § 114, since the summand depends not only on x_i but on Δx as well. But the infinitesimals $\sqrt{1 + \frac{\Delta y_i^2}{\Delta x^2}}\, \Delta x$ and $\sqrt{1 + y_i'^2}\, \Delta x$ differ from each other by an infinitesimal of higher order, since the limit of their ratio is evidently 1, and hence the latter may be substituted for the former, by § 119. Therefore

$$\int_C ds = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + y_i'^2}\, \Delta x$$

$$= \int_a^b \sqrt{1 + y'^2}\, dx,$$

by the theorem of § 114.

7. Find the length of the curve $y = \dfrac{x^4 + 3}{6x}$, from the minimum point to $x = 2$. *Ans.* $\frac{17}{12}$.

8. Rectify the curve $6xy = y^4 + 3$ from $y = 1$ to $y = 4$. *Ans.* $\frac{87}{5}$.

9. Trace the curve $9ay^2 = x(x - 3a)^2$, and find the circumference of the loop. *Ans.* $4\sqrt{3}\,a$.

10. Trace the curve $9y^2 = x^2(2x + 3)$, and find the circumference of the loop. *Ans.* $2\sqrt{3}$.

11. Find the length of the curve $y = \frac{4}{5}x^{\frac{5}{4}}$ between the origin and the point $x = 4$. *Ans.* $\frac{8}{15}(1 + 6\sqrt{3})$.

12. Find the length of the curve $y = \log x$ from $x = 1$ to $x = 2\sqrt{2}$. *Ans.* 2.1.

13. Rectify the curve $y = \log \cos x$ from $x = 0$ to $x = \dfrac{\pi}{4}$. *Ans.* $\log(1 + \sqrt{2})$.

14. Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $8a$.

15. Find the length of a parabola from the vertex to the end of the latus rectum. *Ans.* $2.295\,a$.

16. Rectify the curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ from $t = 0$ to $t = \pi$. *Ans.* $\frac{1}{2}\pi^2 a$.

122. Surfaces of revolution. To find the area of the surface generated by the rotation of a plane curve C about

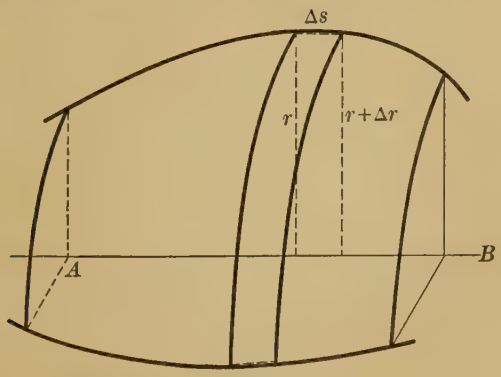


FIG. 73

a line in its plane, let us inscribe in C a broken line of n segments $\Delta s_i'$. Each of the segments $\Delta s_i'$ generates in the

rotation the frustum of a right circular cone, the radii of whose bases may be called r_i and $r_i + \Delta r_i$. The surface of this conical frustum is, by elementary geometry, *the circumference of the middle section multiplied by the slant height*, or $2\pi(r_i + \frac{1}{2}\Delta r_i)\Delta s_i'$. The sum of these surfaces

$$\sum_{i=1}^{n} 2\pi(r_i + \tfrac{1}{2}\Delta r_i)\,\Delta s_i'$$

approaches as its limit the required surface of revolution. Hence

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i\,\Delta s_i' = 2\pi \int_C r\,ds.$$

To evaluate this integral, we replace r and ds by their values in terms of x, y or a suitable parameter (cf. § 121).

Example: Find the surface of a paraboloid of revolution bounded by a right section through the focus.

Given the equation of the generating parabola $y^2 = 4ax$, we have

$$\begin{aligned} S &= 2\pi \int_C y\,ds = 2\pi \int_0^a y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_0^a y\sqrt{1 + \frac{4a^2}{y^2}}\,dx = 2\pi \int_0^a \sqrt{y^2 + 4a^2}\,dx \\ &= 2\pi \int_0^a \sqrt{4ax + 4a^2}\,dx = \frac{4\pi}{a} \cdot \frac{2}{3}(ax + a^2)^{\frac{3}{2}}\Big]_0^a \\ &= \tfrac{8}{3}\pi a^2(2\sqrt{2} - 1). \end{aligned}$$

EXERCISES

1. Work the above example, using y as the variable of integration.

2. Find the surface of a sphere.

3. Find the surface of a sphere, using polar coördinates. (Note that $ds = a\,d\theta$.)

4. Find the surface of a zone cut from a sphere by two parallel planes at a distance h apart. *Ans.* $2\pi ah$.

5. Find the surface generated by revolving the curve $a^2y = x^3$ about Ox, from $x = 0$ to $x = a$. *Ans.* $\frac{1}{27}\pi a^2(10\sqrt{10} - 1)$.

6. Find the surface formed by revolving the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about Ox. *Ans.* $\frac{12}{5}\pi a^2$.

7. Solve Ex. 6, using the equations $x = a\cos^3\phi$, $y = a\sin^3\phi$.

8. Solve the example of § 122, using the equations $x = a\tan^2\phi$, $y = 2a\tan\phi$.

9. Find the surface formed by revolving the catenary $y = a\cosh\frac{x}{a}$ about Oy, from $x = 0$ to $x = a$. *Ans.* $2\pi a^2\left(1 - \frac{1}{e}\right)$.

10. Find the surface of a torus. *Ans.* $4\pi^2ab$.

11. Find the surface generated by revolving about Ox the arc of the curve $y = \frac{x^4 + 3}{6x}$ between the minimum point and the point $x = 2$. *Ans.* $\frac{47}{16}\pi$.

12. Find the surface generated by revolving the arc of Ex. 11 about Oy. *Ans.* $\frac{1}{4}\pi(15 + 4\log 2)$.

13. Find the entire surface generated by revolving the curve $8a^2y^2 = x^2(a^2 - x^2)$ about Ox. *Ans.* $\frac{1}{2}\pi a^2$.

14. Find the surface cut from a sphere by a circular cone of half-angle α with its vertex at the center of the sphere. (Use polar coördinates.)

15. Find the surface generated by revolving an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base. *Ans.* $\frac{64}{3}\pi a^2$.

16. Find the surface generated by revolving about Oy that part of the curve $y = \log x$ lying in the fourth quadrant.
Ans. $\pi[\sqrt{2} + \log(1 + \sqrt{2})]$.

17. Find the surface generated by revolving one arch of the sine curve about Ox. *Ans.* $2\pi[\sqrt{2} + \log(1 + \sqrt{2})]$.

123. Force. Although every physical mass occupies a certain volume, or three-dimensional portion of space, nevertheless it is frequently desirable to introduce the idea of the *material particle*, or geometric point endowed with mass.

If a particle of mass m moves with an acceleration j, the product of the mass by the acceleration is called *force:*

$$F = mj,$$

and the motion is said to be due to the action of the force.

Since force is a mere numerical multiple of acceleration, it follows that force is a *vector* (§ 67). If several forces act on the same particle, their combined effects are equivalent

to that of a single force, their *resultant*. Usually the resultant is most easily found analytically by resolving each force into components parallel to the coördinate axes and summing in each direction to get the rectangular components of the resultant, after which the resultant is found by compounding these rectangular components (see the example below).

If there is no force acting on the particle, or, what is the same thing, if the resultant of all the forces is 0, the particle is said to be *in equilibrium*. It follows from § 69 that a particle in equilibrium is either at rest or moving uniformly in a straight line.

If several forces act at various points of a body, it is not always possible to compound them into a single resultant. In this book, we shall consider only cases in which this is possible.

More generally, we have frequently to consider a force not acting at a single point, but distributed over an area or throughout a volume. Examples are the pressure of a carload of sand against the sides of the car, the attraction between two electrified plates, the gravitational attraction between two spheres or other solids. If the mass upon which the force acts be thought of as composed ultimately of particles, such a distributed force may be regarded as comprising the totality of forces acting on the separate particles.

Of the various problems that arise in this connection, we will consider only one — that of determining the force due to the pressure of a fluid against a vertical retaining wall.

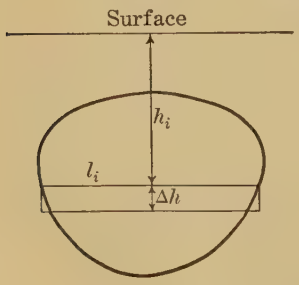


FIG. 74

124. Fluid pressure. Given a plane area A submerged vertically in a fluid, let us divide the area into rectangles of length l_i, width Δh, and depth h_i below the surface

of the fluid, as in Fig. 74. Now the force on any submerged *horizontal* area is equal to the weight of the column of fluid standing on this area. Hence the force on our elementary rectangle is, apart from infinitesimals of higher order,

$$\Delta F = w h_i l_i \Delta h,$$

where w is the weight of the fluid per unit volume*. Hence, by familiar argument, the total force is

$$F = \lim_{\Delta h \to 0} \sum w h_i l_i \Delta h = w \int hl \, dh,$$

where limits of integration are to be assigned in such a way as to extend the integration over the whole area.

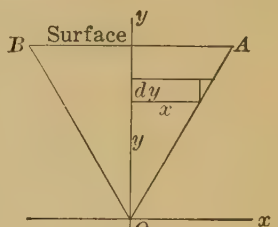


Fig. 75

Example: A trough whose cross-section is an equilateral triangle of side 2 ft. is full of water. Find the force on one end.

Let us take the origin at the lower vertex of the triangle. Then the equation of the line OA is

$$y = \sqrt{3}\, x.$$

The total force on the triangle is

$$F = 2w \int_0^{\sqrt{3}} (\sqrt{3} - y)x \, dy = 2w \int_0^1 (\sqrt{3} - \sqrt{3}\, x)x\sqrt{3} \, dx$$

$$= 6w \int_0^1 (1 - x)x \, dx = 6w \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= w = 62 \text{ lbs., nearly.}$$

* The reasoning, in detail, is as follows. If the rectangle were rotated about its upper side through 90° into a horizontal position at the depth h_i, the force would be $wh_il_i\Delta h$; if it were rotated about the lower side through 90°, thus assuming a horizontal position at the depth $h_i + \Delta h$, the force would be $w(h_i + \Delta h)l_i\Delta h$. The actual force is evidently greater than the former value and less than the latter.

EXERCISES

1. Find the force on one face of a plank 2×8 ft. submerged vertically with its upper end (*a*) in the surface, (*b*) 4 ft. below the surface.
Ans. (*a*) $64\,w$; (*b*) $128\,w$.

2. What force must be withstood by a vertical dam 100 ft. long and 20 ft. deep? *Ans.* 620 tons.

3. Find the force on one face of a right triangle of sides $AB = 3$ ft., $AC = 4$ ft., submerged with AC vertical and A in the surface.
Ans. $8w$.

4. Work Ex. 3 if A is 2 ft. below the surface. *Ans.* $20\,w$.

5. Work Ex. 3 if A is 6 ft. and C 2 ft. below the surface.
Ans. $28\,w$.

6. What force must be withstood by a trapezoidal dam 100 ft. long at the top, 80 ft. long at the bottom, and 20 ft. deep? *Ans.* 17,333 w.

7. A horizontal cylindrical boiler 4 ft. in diameter is half full of water. Find the force on one end. *Ans.* 330 lbs.

8. Work Ex. 7 if the boiler is full of water. *Ans.* $8\,\pi w$.

9. Find the force that must be withstood by a bulkhead closing a water main 4 ft. in diameter, if the surface of the water in the reservoir is 40 ft. above the center of the bulkhead. *Ans.* 16 tons.

10. A trough 4 ft. deep and 6 ft. wide has semi-elliptical ends. If the trough is full of water, find the force on one end. *Ans.* $32\,w$.

11. Solve Ex. 10, using parametric equations.

12. Find the force on one face of a square 2 ft. on a side, submerged with one diagonal vertical and one corner in the surface. *Ans.* $4\sqrt{2}\,w$.

13. Find the force on one end of a parabolic trough full of water, if the depth is 2 ft. and the width across the top 2 ft. *Ans.* $\frac{32}{15}\,w$.

CHAPTER XIX

CENTROIDS

125. Density. A mass is said to be *homogeneous* if the masses contained in any two equal volumes are equal. In all other cases the mass is *heterogeneous*. In the present chapter we confine our attention to homogeneous masses.

The *density* of a homogeneous mass is the ratio of the mass M to the volume V that it occupies. That is, the density is the mass per unit volume.

The mass-point (§ 123) may be imagined as the limiting form approached by a body whose dimensions approach 0, while the density increases in such a way that the mass remains finite. Similarly we may think of masses of one dimension and of two dimensions — *i.e.* of material curves and surfaces. Such masses are represented approximately, for example, by slender wires and thin sheets of metal. In these cases we define the density as "linear density," or mass per unit length, and "surface density," or mass per unit area.

126. Moment of mass. The product of a mass m, concentrated at a point P, by the distance l of P from a given point, line, or plane, is called the *mass-moment* * of m with respect to the point, line, or plane. Denoting this moment by G, we have

$$G = ml.$$

If a system of points $P_1, P_2, \cdots, P_n$, having masses $m_1, m_2, \cdots, m_n$ respectively, be referred to cartesian coördinate axes, the moments of the system with respect to the three

* Also called the *simple moment*, or *moment of first order*.

coördinate planes are respectively

$$G_{yz} = \sum_{i=1}^{n} m_i x_i, \quad G_{zx} = \sum_{i=1}^{n} m_i y_i, \quad G_{xy} = \sum_{i=1}^{n} m_i z_i.$$

In case the particles all lie in one of the coördinate planes, the moments with respect to coördinate planes reduce to moments with respect to coördinate axes.

The idea of mass-moment may be extended to the case of a continuous mass by thinking of the mass as composed of an indefinitely large number of particles. The actual computation of such a moment is usually effected by means of definite integrals; we return to this question presently.

127. Centroid. Given any mass M, let G_{yz}, G_{zx}, G_{xy} denote the moments of the mass with respect to the coördinate planes. The point C whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, are given by the formulas

$$M\bar{x} = G_{yz}, \quad M\bar{y} = G_{zx}, \quad M\bar{z} = G_{xy}$$

clearly has the property that the moment of the mass with respect to each of the coördinate planes is the same as if the whole mass were concentrated at that point.

It is easily shown that this property holds for moments with respect to any other plane. The proof for the general case requires the use of multiple integrals (Chapter XXIV); for a system of particles the proof is as follows. Let

$$ax + by + cz = p \tag{1}$$

be the equation of any plane in the normal form; let $\bar{p}$, p_1, p_2, $\cdots$, p_n be the distances of the points C, P_1, P_2, $\cdots$, P_n from this plane. Now

$$\begin{aligned} p_1 &= ax_1 + by_1 + cz_1 - p, \\ &\vdots \\ p_n &= ax_n + by_n + cz_n - p, \end{aligned}$$

so that

$$\begin{aligned} \sum_{i=1}^{n} m_i p_i &= a \sum_{i=1}^{n} m_i x_i + b \sum_{i=1}^{n} m_i y_i + c \sum_{i=1}^{n} m_i z_i - p \sum_{i=1}^{n} m_i \\ &= aM\bar{x} + bMy + cM\bar{z} - Mp = M\bar{p}. \end{aligned}$$

That is, the moment of the system with respect to the plane (1) is the same as if the whole mass were concentrated at C.

The point C is called the *center of mass*, or *centroid* *:

The centroid of a mass is a point such that the moment of the mass with respect to any plane is the same as if the whole mass were concentrated at that point.

By § 126, the coördinates of the centroid of a *system of particles* are given by the formulas

$$M\bar{x} = \sum_{i=1}^{n} m_i x_i, \quad M\bar{y} = \sum_{i=1}^{n} m_i y_i, \quad M\bar{z} = \sum_{i=1}^{n} m_i z_i.$$

The centroid evidently has the following important property:

The moment of a mass with respect to any plane through the centroid is 0.

In the actual determination of centroids, the following considerations are often useful (the first two apply only to homogeneous masses):

(*a*) *If the body has a geometrical center, that point is the centroid.*

(*b*) *Any plane or line of symmetry must contain the centroid.*

(*c*) *If the body consists of several portions for each of which the centroid can be found, each portion may be imagined concentrated at its centroid;* the problem thus reduces to the consideration of a set of particles.

It is clear that, for a homogeneous body of given size and shape, both the mass and its moment with respect to any plane are proportional to the density δ. Hence, in the formulas for $\bar{x}$, $\bar{y}$, $\bar{z}$, the factor δ cancels out from both members, leaving the coördinates of the centroid independent of the density. We may therefore without loss of generality take $\delta = 1$, and are thus led to speak of centroids of geometrical figures — volumes, areas, and lines — without reference to the idea of mass.

* The centroid coincides with the "center of gravity."

EXERCISES

Find the centroid of each of the following plane systems of particles.

1. Equal particles at (0, 0), (1, 3), (2, − 5). *Ans.* $(1, -\frac{2}{3})$.

2. Equal particles at (3, 0), (− 2, − 4), (1, 5), (0, 20). *Ans.* $(\frac{1}{2}, \frac{21}{4})$.

3. Masses of 1, 2, 3 units at (3, 1), (− 2, 1), (1, 4) respectively. *Ans.* $(\frac{1}{3}, \frac{5}{2})$.

4. Masses of 3, 2, 5 units at (0, 2), (2, 4), (2, − 3) respectively. *Ans.* $(\frac{7}{5}, -\frac{1}{10})$.

5. Show that the centroid of two particles divides the line joining them into segments inversely proportional to the masses.

6. Show that the centroid of three equal particles lies at the intersection of the medians of the triangle having the three points as vertices.

7. Equal particles are placed at five of the six vertices of a regular hexagon. Find the centroid. *Ans.* $\bar{x} = \frac{1}{6}a$.

8. Equal particles are placed at the points (0, 2, 1), (1, 5, 3), (− 1, − 2, 2). Find the centroid. *Ans.* $(0, \frac{5}{3}, 2)$.

9. Masses of 1, 2, 3, 4 units are placed at the points (0, − 2, 0), (0, 0, 2), (4, − 1, 5), (− 3, 0, − 1) respectively. Find the centroid. *Ans.* $(0, -\frac{1}{2}, \frac{3}{4})$.

10. A slender rod 12 ft. long is bent at the midpoint so as to form a right angle. Find the centroid. *Ans.* $(\frac{3}{2}, \frac{3}{2})$.

11. Solve Ex. 10 if the right angle occurs at a point of trisection. *Ans.* $(\frac{8}{3}, \frac{2}{3})$.

12. Two posts of equal radius, 6 ft. and 4 ft. long, stand upright 2 ft. apart. Find the centroid. *Ans.* $(\frac{4}{5}, \frac{13}{5})$.

13. Three rods of lengths 10, 10, and 12 ft. form a triangle. Find the centroid. *Ans.* $\bar{y} = \frac{5}{2}$ ft.

14. Weights of 10, 20, 20, and 30 lbs. rest on the four corners of a table 6 ft. square. Find the centroid of the weights. *Ans.* $(3, \frac{15}{4})$.

15. From one corner of a sheet of paper 5 × 7 in. a square of side 2 in. is cut out. Find the centroid of the remainder. *Ans.* (2.31, 3.18).

16. Find the centroid of an angle-iron, the width of the flanges being 2 in. and the thickness $\frac{1}{4}$ in. *Ans.* $(\frac{71}{120}, \frac{71}{120})$.

17. Solve Ex. 16 if one flange is 3 in. wide and $\frac{1}{2}$ in. thick, the other 2 in. wide and $\frac{1}{4}$ in. thick. *Ans.* $(\frac{83}{140}, \frac{9}{20})$.

18. The legs of a table are 2 ft. 6 in. long; the legs weigh 5 lbs. each, the top 20 lbs. Find the centroid. *Ans.* $\bar{z} = \frac{15}{8}$.

19. Solve Ex. 18 if the table is 6×4 ft., and a weight of 20 lbs. rests on one corner. *Ans.* $(2, \frac{4}{3}, \frac{25}{12})$.

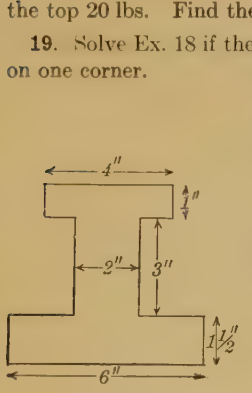


Fig. 76

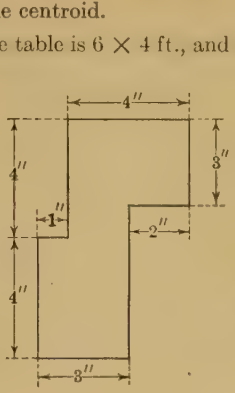


Fig. 77

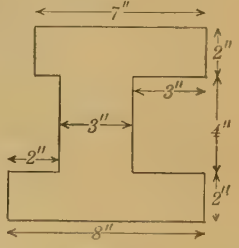


Fig. 78

20. Find the centroid of the area in Fig. 76. *Ans.* $\bar{y} = 2.36$.

21. Find the centroid of the area in Fig. 77. *Ans.* (2.23, 4.27).

22. Find the centroid of the area in Fig. 78. *Ans.* (4.02, 3.86).

23. The base of a box is 4×3 ft., the depth is 2 ft. If there is no top, find the centroid. *Ans.* $\bar{y} = 0.7$ ft.

24. Solve Ex. 23 if the bottom is twice as heavy, per unit area, as the sides. *Ans.* $\bar{y} = 0.54$ ft.

25. From a circular disk a round hole is punched out, the two circles being tangent internally. Find the centroid of the remaining figure.

26. From a circular plate of radius 4 in. a hole 2 in. square is cut out, one corner of the square being at the center of the plate. Find the centroid of the remainder. *Ans.* $\bar{x} = -0.12$ in.

27. Find the centroid of a cylindrical basin of radius 4 in. and depth 3 in. *Ans.* $\bar{y} = \frac{9}{10}$ in.

28. Solve Ex. 27 if the bottom is twice as heavy, per unit area, as the sides. *Ans.* $\bar{y} = \frac{9}{14}$ in.

29. Find the centroid of two spheres that are tangent to each other internally.

30. A sphere of radius 3 in. rests on a cylinder of radius 4 in. and height 8 in. Find the centroid. *Ans.* $\bar{y} = 5.54$ in.

31. A monument is composed of a block of stone of base 4×3 ft. and height 2 ft. 6 in., surmounted by a cube of side 2 ft., this in turn supporting a sphere of radius 1 ft. Find the centroid of the whole figure. *Ans.* $\bar{y} = 2.10$ ft.

128. Determination of centroids by integration. To find the centroid of a continuous mass, we must in general resort to integration. In the most general case multiple integrals (Chapter XXIV) must be used, but in most cases of practical importance the result may be obtained by methods analogous to those of Chapter XVIII.

In the following discussion we restrict ourselves to one-, two-, or three-dimensional bodies of the forms considered in Chapter XVIII*. Let us choose, as in that chapter, a suitable geometrical element (of volume, area, or length), and denote the mass contained in this element by Δm_i. Let x_i, y_i, z_i be the coördinates of the *centroid* † of Δm_i. Then the sum $\sum_{i=1}^{n} x_i \Delta m_i$ represents approximately the moment of the mass with respect to the yz-plane (or the y-axis, in the case of a plane mass in the xy-plane), and the limit of this sum as n becomes infinite is exactly the moment in question. In this way we obtain the following formulas for the coördinates of the centroid:

$$M\bar{x} = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \Delta m_i,$$

$$M\bar{y} = \lim_{n\to\infty} \sum_{i=1}^{n} y_i \Delta m_i,$$

$$M\bar{z} = \lim_{n\to\infty} \sum_{i=1}^{n} z_i \Delta m_i.$$

Now upon recalling the meaning of Δm_i, we see that in any given case the above limits may be expressed as definite integrals by the fundamental theorem of § 114. The result may be written in the following form:

$$(1) \qquad M\bar{x} = \int x\, dm, \quad M\bar{y} = \int y\, dm, \quad M\bar{z} = \int z\, dm,$$

* Nevertheless the formulas obtained are applicable, with proper interpretation, to *all* masses.

† It follows from the theorem of § 119 that in the expressions for these coördinates, *all infinitesimals may be neglected.*

where x, y, z are the coördinates of the centroid of the mass-element. In any problem each integrand must be expressed in terms of a single variable, and limits are to be assigned in such a way as to extend the integration over the whole mass.

In the next few articles we explain more in detail the application of the above formulas.

129. Centroids of plane areas. To find the centroid of a plane area we choose an element of area as in § 115 (or § 116, if polar coördinates are used), and build up the moment-integrals by the formulas (1) of § 128.

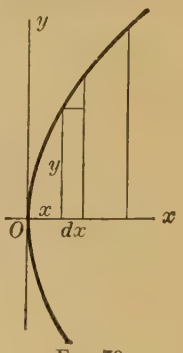


FIG. 79

Example: Find the centroid of the area in the first quadrant bounded by the parabola $y^2 = 4ax$ and its latus rectum.

With the element chosen as in Fig. 79, the area is

$$A = \int_0^a y\,dx = \tfrac{4}{3}a^2.$$

The centroid of the element is obviously the point $(x, \frac{1}{2}y)$; hence we have

$$A\bar{x} = \int_0^a xy\,dx = 2\sqrt{a}\int_0^a x^{\frac{3}{2}}\,dx = \tfrac{4}{5}a^3,$$

$$A\bar{y} = \int_0^a \tfrac{1}{2}y \cdot y\,dx = 2a\int_0^a x\,dx = a^3.$$

Therefore

$$\bar{x} = \tfrac{3}{5}a,\ \bar{y} = \tfrac{3}{4}a.$$

We may also find $\bar{y}$ very easily by taking the element parallel to Ox. Thus

$$A\bar{y} = \int_0^{2a} y(a - x)\,dy = 2a\int_0^a (a - x)\,dx = a^3, \text{ etc.}$$

EXERCISES

Find the centroid of each of the following areas. In each case, draw a figure and estimate the coördinates of the centroid, thus obtaining a rough check on the result.

1. A right triangle. *Ans.* At the intersection of the medians.

2. A semicircular area. Solve in two ways. *Ans.* $\bar{x} = \dfrac{4\,a}{3\,\pi}$.

3. One quadrant of an ellipse. Solve in two ways.

Ans. $\left(\dfrac{4\,a}{3\,\pi}, \dfrac{4\,b}{3\,\pi}\right)$.

4. One quadrant of an ellipse, using the equations $x = a\cos\phi$, $y = b\sin\phi$.

5. Any triangle. *Ans.* At the intersection of the medians.

6. The area under the hyperbola $xy = a^2$, from $x = a$ to $x = 2\,a$.

7. The area of Ex. 6, using the equations $x = a\sec\phi$, $y = a\cos\phi$.

8. Half an arch of the sine curve. *Ans.* $(1, \frac{1}{8}\pi)$.

9. Half an arch of the curve $y = 2\cos\frac{1}{3}x$. *Ans.* $(\frac{3}{2}\pi - 3, \frac{1}{4}\pi)$.

10. The area under the curve $y = e^x$, from $x = 0$ to $x = 1$.

Ans. (0.58, 0.93).

11. The area in the second quadrant under the curve $y = e^x$. Obtain each coördinate in two ways. *Ans.* $(-1, \frac{1}{4})$.

12. The area bounded by the curve $y = x^2$, the x-axis, and the line $x = 3$. Find $\bar{y}$ in two ways. *Ans.* $(\frac{9}{4}, \frac{27}{10})$.

13. The area bounded by the parabolas $y^2 = 4\,ax$, $x^2 = 4\,ay$.

14. Any quadrilateral. Solve graphically.

15. The area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. *Ans.* $(\frac{1}{5}\,a, \frac{1}{5}\,a)$.

16. The area between the curves $2\,y = x^2$, $y = x^3$. *Ans.* $(\frac{3}{10}, \frac{3}{70})$.

17. The area bounded by the curve $y = \dfrac{\log x}{x}$, the x-axis, and the maximum ordinate. *Ans.* $\left(2, 2 - \dfrac{5}{e}\right)$.

18. One arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

Ans. $(\pi a, \frac{5}{6}\,a)$.

19. The area in the first quadrant between an ellipse and its major auxiliary circle. *Ans.* $\left(\dfrac{4\,a}{3\,\pi}, \dfrac{4\,a + 4\,b}{3\,\pi}\right)$.

20. The area between a quadrant of a circle and the tangents at its extremities. *Ans.* $\left(\dfrac{2\,a}{12 - 3\,\pi}, \dfrac{2\,a}{12 - 3\,\pi}\right)$.

21. The area between a quadrant of a circle and the chord joining its extremities. *Ans.* $\left(\frac{2a}{3\pi - 6}, \frac{2a}{3\pi - 6}\right)$.

FIG. 80

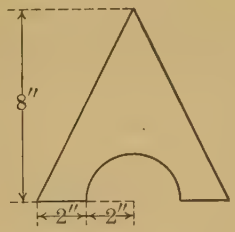


FIG. 81

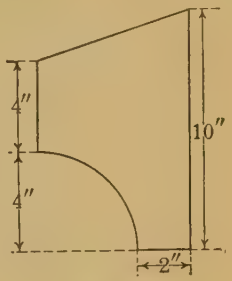


FIG. 82

22. The area in Fig. 80. *Ans.* $\bar{y} = 1.02$.

23. The area in Fig. 81. *Ans.* $\bar{y} = 3.11$.

24. The area in Fig. 82. *Ans.* (3.54, 5.37).

25. A semicircular area, using polar coördinates.

26. A circular sector. *Ans.* $\bar{x} = \frac{2}{3} a \frac{\sin \alpha}{\alpha}$.

27. Half of a circular ring. *Ans.* $x = \frac{4}{3\pi} \cdot \frac{a^2 + ab + b^2}{a + b}$.

28. Half the area of the hypocycloid $x = a \cos^3 \phi$, $y = a \sin^3 \phi$. *Ans.* $\bar{x} = \frac{256\, a}{315\, \pi}$.

29. The area bounded by the curves $y = x$, $y = 2x$, $y = x^2$, using polar coördinates. *Ans.* $(\frac{15}{14}, \frac{8}{35})$.

30. Prove the *second proposition of Pappus:*

The volume of any solid of revolution is equal to the product of the generating area into the circumference of the circle described by the centroid of the area.

Solve the following by using the second proposition of Pappus.

31. Find the volume of a torus. *Ans.* $2\pi^2 a^2 b$.

32. Find the centroid of a right triangle.

33. Find the centroid of a semicircular area.

34. Find the volume generated by revolving an ellipse about the tangent at a vertex.

35. Find the centroid of half of a circular ring. (Cf. Ex. 27.)

36. The triangle bounded by the lines $x = 0$, $y = a$, $x + y = 2a$ revolves about Ox. Find the volume generated.

130. Centroids of solids. The centroid of a solid of revolution evidently lies on the axis of revolution, so that a single coördinate determines its position. In applying the formulas of § 128 we may of course choose as an element either a circular disk or a cylindrical shell, according to convenience.

In certain cases the centroids of other solids may be found by simple integration (cf. Exs. 17–36 below).

EXERCISES

Find the centroid of each of the following volumes. Draw a figure in each case and estimate the coördinates of the centroid.

1. A hemisphere. Solve in two ways. *Ans.* $\bar{x} = \frac{3}{8}a$.

2. A right circular cone. *Ans.* $\bar{x} = \frac{1}{4}h$.

3. A paraboloid of revolution bounded by a right section through the focus. *Ans.* $\bar{x} = \frac{2}{3}a$.

4. Half an ellipsoid of revolution. Solve in various ways. *Ans.* $\bar{x} = \frac{3}{8}a$.

5. The volume formed by rotating about the y-axis the area under the parabola $y^2 = 4ax$ from $x = 0$ to $x = a$. *Ans.* $\bar{y} = \frac{2}{5}a$.

6. The volume formed by rotating the area of Ex. 5 about the latus rectum. *Ans.* $\bar{y} = \frac{5}{8}a$.

7. The volume formed by rotating about Ox the area bounded by the hyperbola $x^2 - y^2 = a^2$, the x-axis, and the line $x = 2a$. *Ans.* $\bar{x} = \frac{27}{16}a$.

8. The volume formed by rotating the area of Ex. 7 about Oy. *Ans.* $\bar{y} = \frac{3}{8}\sqrt{3}\,a$.

9. The volume of Ex. 8, using the equations $x = a\sec\phi$, $y = a\tan\phi$.

10. The volume generated by revolving about Ox the area in the second quadrant under the curve $y = e^x$. *Ans.* $\bar{x} = -\frac{1}{2}$.

11. The volume generated by revolving about Oy the area in the second quadrant under the curve $y = e^x$. Solve in two ways.

12. The volume of Ex. 44, p. 191. *Ans.* $\bar{z} = \frac{2}{3}c$.

13. The volume of Ex. 46, p. 191. *Ans.* $\bar{x} = \frac{12}{5}a$.

14. The volume of Ex. 48, p. 191. *Ans.* $\bar{z} = 2a$.

15. Half the volume of Ex. 49, p. 191. *Ans.* $\bar{z} = \frac{3}{8}c$.

16. The volume formed by revolving about the y-axis the area under the curve $y = e^{-\frac{1}{2}x^2}$. Evaluate the moment-integral in two ways. *Ans.* $\bar{y} = \frac{1}{4}$.

17. A tetrahedron with three mutually perpendicular faces.
Ans. $(\frac{1}{4}a, \frac{1}{4}b, \frac{1}{4}c)$.

18. An elliptic cone. *Ans.* $\bar{x} = \frac{1}{4}h$.

19. A pyramid with a square base, whose apex lies vertically above one corner of the base. *Ans.* $(\frac{3}{8}a, \frac{3}{8}a, \frac{1}{4}h)$.

20. One eighth of the volume common to two equal circular cylinders whose axes intersect at right angles. (Cf. Ex. 12, p. 193.)
Ans. $(\frac{9}{64}\pi a, \frac{9}{64}\pi a, \frac{3}{8}a)$.

21. The volume inclosed by the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ and the planes $z = 0$, $z = c$. *Ans.* $\bar{z} = \frac{9}{16}c$.

22. One half the volume in the example of § 120. Find each coördinate in two ways. *Ans.* $(\frac{3}{4}, \frac{3}{8}\pi, \frac{3}{16}\pi)$.

23. The volume in the first octant under the plane $z = y$ and inside the cylinder $y^2 = 4 - x$. Solve in two ways. (Cf. Ex. 15, p. 193.)
Ans. $(\frac{4}{3}, \frac{16}{15}, \frac{8}{15})$.

24. The volume in Ex. 16, p. 193.

25. The volume in Ex. 17, p. 193.

26. The volume in Ex. 18, p. 193.

27. The volume in Ex. 20, p. 193. *Ans.* $(\frac{1}{3}b, \frac{2}{5}a, \frac{1}{3}a)$.

28. The volume in Ex. 23, p. 193. *Ans.* $(\frac{1}{3}a, \frac{2}{3}b, \frac{2}{5}b)$.

29. The volume in Ex. 24, p. 193.

30. One quarter of a right circular conoid. (Ex. 25, p. 194.)
Ans. $\left(\frac{8}{9}\frac{a}{\pi}, \frac{h}{3}, \frac{4}{3}\frac{a}{\pi}\right)$.

31. The volume in Ex. 31, p. 194. *Ans.* $(\frac{4}{7}, \frac{3}{4}, \frac{2}{5})$.

32. The volume in Ex. 33, p. 194. *Ans.* $(\frac{3}{8}a, \frac{3}{32}\pi a, \frac{3}{32}\pi a)$.

33. The volume in Ex. 34, p. 194. *Ans.* $(\frac{3}{16}\pi a, \frac{3}{32}\pi a, \frac{3}{8}b)$.

34. The volume in Ex. 35, p. 194.

35. The volume in Ex. 36, p. 194.

36. The volume in Ex. 37, p. 194.

37. A top consists of a cone of radius 2 in. and height 2 in., surmounted by a cylinder of radius $\frac{1}{4}$ in. and height 2 in. Find the centroid. *Ans.* 1.57 in. from vertex of cone.

38. A buoy consists of a cone and hemisphere placed base to base; the common radius is 2 ft., the height of the cone 6 ft. Find the centroid. *Ans.* 5.4 ft. from vertex.

39. Find the centroid of the volume formed by rotating the area in Fig. 81, p. 211, about its axis of symmetry. *Ans.* $\bar{y} = \frac{9}{25}\frac{1}{5}$.

131. Centroids of lines; of curved surfaces. The centroid of a curved arc or of a surface of revolution can be found by choosing an element as in § 121 or § 122 respectively.

Example: Find the centroid of a semicircular wire.

Taking the bounding diameter as axis of y, we have

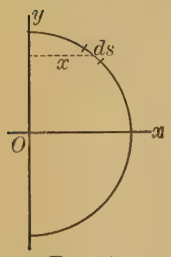


FIG. 83

$$s\bar{x} = \int_c x\,ds = 2\int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^a x\sqrt{1+\frac{x^2}{y^2}}\,dx = 2a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}$$

$$= -2a\sqrt{a^2-x^2}\Big]_0^a = 2a^2.$$

Hence

$$\bar{x} = \frac{2a^2}{s} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}.$$

EXERCISES

Find the following centroids.

1. A semicircular wire, using polar coördinates.
2. A circular arc.
3. Half the arc of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\bar{x} = \frac{2}{5}a$.
4. Half the arc of the hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$.
5. A hemispherical surface. *Ans.* $\bar{x} = \frac{1}{2}a$.
6. A hemispherical surface, using polar coördinates.
7. A spherical zone. *Ans.* Midway between the bounding planes.
8. The lateral surface of a right circular cone. *Ans.* $\bar{x} = \frac{1}{3}h$.
9. The surface cut from a sphere by one sheet of a cone of half-angle 60°. *Ans.* $\bar{x} = \frac{3}{4}a$.
10. Find the centroid of a trough whose cross-section is a semicircle of radius 1 ft. *Ans.* $\bar{z} = 0.383$.
11. Prove the *first proposition of Pappus:*

The surface of a solid of revolution is equal to the length of the generating arc times the circumference of the circle described by the centroid of the arc.

Solve the following by the first proposition of Pappus.

12. Find the surface of a right circular cone.
13. Find the surface of a torus.
14. Find the centroid of a semicircular wire.

CHAPTER XX

MOMENTS OF INERTIA

132. Moment of inertia. The product of a mass m, concentrated at a point P, by the square of the distance r of P from a fixed line, or *axis*, is called the moment of the second order, or *moment of inertia*, of m with respect to the given axis:

$$I = mr^2.$$

The moment of inertia of a system of such masses is of course the sum

$$I = \sum_{i=1}^{n} m_i r_i^2.$$

If we think of a "continuous" mass as composed ultimately of particles, the meaning of moment of inertia of such a mass becomes clear.

The moment of inertia of a homogeneous body is proportional to the density δ. Taking $\delta = 1$, we may speak of "moment of inertia" of areas, volumes, etc., no idea of mass being involved.

133. Radius of gyration. The moment of inertia of any mass M may always be written in the form

$$I = MR^2,$$

where the quantity R is called the *radius of gyration*, or *radius of inertia*, of M with respect to the given axis of moments. The meaning of the radius of gyration is obvious: it is the distance from the axis at which a *particle* of mass M must be placed in order to have the same moment of inertia as the original mass.

134. Determination of moment of inertia by integration. The actual computation of the moment of inertia of a continuous mass is effected by integration in much the same way that the moment of the first order (§ 128) is determined. In the general case, we must have recourse to double or triple integrals (Chapter XXIV); but for the simple cases of practical importance the result can usually be found by a single integration.

For the present we consider only such bodies as were studied in Chapter XVIII. Let us choose an element (of volume, area, or length) in some suitable way, and denote the mass of this element by Δm_i. Let r_i denote the radius of gyration * of the element. Then the sum $\sum_{i=1}^{n} r_i^2 \Delta m_i$ is approximately the moment of inertia of the mass with respect to the given axis, and the limit of this sum is exactly the required moment:

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} r_i^2 \Delta m_i.$$

Now if we apply the theorem of § 114, the above limit appears as the definite integral

$$I = \int r^2 \, dm, \tag{1}$$

where r is the *radius of gyration of the mass element* with respect to the axis of moments. Of course the integrand must be expressed in terms of a single variable, and the integration must be extended over the whole mass.

Just as, in finding centroids, we must take an element the position of whose centroid is known, so here the essential point is to *choose an element whose radius of gyration is known*. Thus the moment of inertia of a plane area with respect to a line in its plane may be found by taking as the element a rectangle parallel to the axis of moments, since then the

* In the expression for the radius of gyration, infinitesimals may as usual be neglected.

radius of gyration of the element is simply its distance from the axis.

To find the moment of inertia of a volume of revolution about the axis of revolution it is usually best to take as element a cylindrical shell (§ 118).

Examples: (*a*) Find the moment of inertia, with respect to the y-axis, of the area bounded by the parabola $y^2 = 4ax$, the x-axis, and the latus rectum (Fig. 84).

Taking the element parallel to Oy, we find

$$I_y = \int_0^a x^2 y\, dx = 2\sqrt{a} \int_0^a x^{\frac{5}{2}}\, dx = \tfrac{4}{7} a^4.$$

Since the mass, or area, is

$$M = \int_0^a y\, dx = \tfrac{4}{3} a^2,$$

we may write

$$I_y = \tfrac{3}{7} Ma^2,$$

which shows that the square of the radius of gyration is

$$R^2 = \tfrac{3}{7} a^2.$$

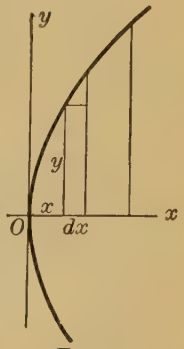


Fig. 84

(*b*) The area in Fig. 84 revolves about the y-axis. Find the moment of inertia of the volume generated, with respect to the axis of revolution.

Take as element of volume the cylindrical shell generated by the rectangle shown in Fig. 84, so that

$$dV = 2\pi xy\, dx.$$

The radius of gyration of this shell about the y-axis is evidently x. Hence

$$I_y = 2\pi \int_0^a x^2 \cdot xy\, dx = 4\pi\sqrt{a} \int_0^a x^{\frac{7}{2}}\, dx = \tfrac{8}{9} \pi a^5.$$

The mass, or volume, is

$$M = 2\pi \int_0^a xy\, dx = \tfrac{8}{5} \pi a^3,$$

whence

$$I_y = \tfrac{5}{9} Ma^2.$$

EXERCISES

Find the following moments of inertia.

1. A mass of 3 units at (0, 0), one of 4 units at (2, 2), and one of 5 units at $(-1, -3)$, with respect to each of the coördinate axes.

Ans. $I_y = 21$, $I_x = 61$.

2. A mass of 2 units at (3, 5), one of 5 units at $(0, -2)$, and one of 6 units at $(4, -4)$, with respect to each coördinate axis.

Ans. $I_y = 114$, $I_x = 166$.

3. Equal particles of mass m at (0, 2, 5), (3, 2, -2), (1, 5, -3), (4, 0, 2), with respect to each coördinate axis.

Ans. $I_x = 75\,m$, $I_y = 68\,m$, $I_z = 59\,m$.

4. Equal particles at each corner of a square, with respect to one side of the square. *Ans.* $\frac{1}{2}Ma^2$.

5. Equal particles at each corner of a cube, with respect to an edge of the cube. *Ans.* Ma^2.

6. A straight rod or wire with respect to a perpendicular through one end. *Ans.* $\frac{1}{3}Ml^2$.

7. A wire bent in the form of a square, with respect to one side.

Ans. $\frac{5}{12}Ma^2$.

8. A rectangle (area) with respect to one side. *Ans.* $\frac{1}{3}Ma^2$.

9. A circle with respect to a diameter. *Ans.* $\frac{1}{4}Ma^2$.

10. (*a*) An isosceles triangle, (*b*) any triangle, with respect to the base. *Ans.* $\frac{1}{6}Mh^2$.

11. A circular disk of radius 4 in. with a square of side 2 in. cut out of the center, with respect to a diameter parallel to a side of the square.

Ans. $64\pi - \frac{4}{3}$.

12. The area in Fig. 84 with respect to Ox, (*a*) taking the element parallel to Ox, (*b*) taking the element parallel to Oy and using the answer to Ex. 8. *Ans.* $\frac{4}{5}Ma^2$.

13. The area in Fig. 84, with respect to the latus rectum.

Ans. $\frac{8}{35}Ma^2$.

14. An ellipse with respect to each of its axes. *Ans.* $\frac{1}{4}Ma^2$, $\frac{1}{4}Mb^2$.

15. An ellipse with respect to each of its axes, using the equations $x = a\cos\phi$, $y = b\sin\phi$.

16. The area in the fourth quadrant under the curve $y = \log x$, with respect to Oy. Solve in two ways. *Ans.* $\frac{1}{6}M$.

17. The area under the hyperbola $xy = a^2$ from $x = a$ to $x = 2a$, with respect to Oy. *Ans.* $\frac{3}{2}a^4$.

18. The area of Ex. 17, with respect to Ox. *Ans.* $\frac{1}{8}a^4$.

19. The area in Fig. 76, p. 207, with respect to the base.
Ans. $165\frac{7}{12}$.

20. The area in Fig. 76, p. 207, with respect to the line of symmetry.
Ans. $34\frac{1}{3}$.

21. The area in Fig. 77, p. 207, with respect to the base.
Ans. $620\frac{2}{3}$.

22. The area in Fig. 78, p. 207, with respect to the base.
Ans. 920.

23. The area in Fig. 80, p. 211, with respect to the base of the triangle.
Ans. $72 + 2\pi$.

24. The area in Fig. 81, p. 211, with respect to the line of symmetry.
Ans. $\frac{256}{3} - 2\pi$.

25. The area in Fig. 81, p. 211, with respect to the base.
Ans. $\frac{1024}{3} - 2\pi$.

26. The area in Fig. 82, p. 211, with respect to the left-hand side.
Ans. $684 - 16\pi$.

27. The area under one arch of the sine-curve, with respect to Ox.
Ans. $\frac{4}{3}$.

In Exs. 28–44, find the moment of inertia with respect to the axis of revolution.

28. A cylinder of revolution, or a circular disk. *Ans.* $\frac{1}{2}Ma^2$.

29. A sphere. Solve by two methods (note the answer to Ex. 28).
Ans. $\frac{2}{5}Ma^2$.

30. A circular cone. *Ans.* $\frac{3}{10}Ma^2$.

31. A paraboloid of revolution bounded by the right section through the focus. Solve by two methods. *Ans.* $\frac{4}{3}Ma^2$.

32. A prolate spheroid. Solve by two methods. *Ans.* $\frac{2}{5}Mb^2$.

33. An oblate spheroid, using the equations $x = a\cos\phi$, $y = b\sin\phi$.
Ans. $\frac{2}{5}Ma^2$.

34. The volume of Ex. 13, p. 189. *Ans.* $\frac{14}{5}Ma^2$.

35. The volume of Ex. 15, p. 189. *Ans.* $\frac{7}{10}Ma^2$.

36. The volume generated by revolving the curve $y = e^{-\frac{1}{2}x^2}$ about the y-axis. Solve in two ways. *Ans.* $2M$.

37. A torus. *Ans.* $(b^2 + \frac{3}{4}a^2)M$.

38. The volume formed by revolving an ellipse about the tangent at a vertex. *Ans.* $\frac{7}{4}Ma^2$.

39. The volume of Ex. 46, p. 191.

40. The volume of Ex. 48, p. 191. *Ans.* $\frac{32}{5}Ma^2$.

41. The volume of Ex. 49, p. 191. *Ans.* $\frac{3}{8} Ma^2$.

42. The volume of Ex. 50, p. 191.

43. The volume formed by revolving about Ox the area under the hyperbola $xy = a^2$ to the right of the line $x = a$. *Ans.* $\frac{1}{6}\pi a^5$.

44. The volume of Ex. 43, using the equations $x = a \sec \phi$, $y = a \cos \phi$.

Find the following moments of inertia.

45. A wire bent in the form of an isosceles triangle, with respect to the line of symmetry. *Ans.* $\frac{2}{3} a^2(a + \sqrt{a^2 + h^2})$.

46. A wire bent in the form of a square, with respect to a diagonal. *Ans.* $\frac{1}{3} Ma^2$.

47. A line segment with respect to an axis through one end making an angle α with it. *Ans.* $\frac{1}{3} l^2 \sin^2 \alpha$.

48. A circular wire with respect to a diameter, using (*a*) polar, (*b*) cartesian coördinates. *Ans.* $\frac{1}{2} Ma^2$.

49. A spherical surface about a diameter, using (*a*) polar, (*b*) cartesian coördinates. *Ans.* $\frac{2}{3} Ma^2$.

50. The lateral surface of a cone of revolution, about its axis. *Ans.* $\frac{1}{2} Ma^2$.

51. The surface of a torus, with respect to its axis. *Ans.* $(b^2 + \frac{3}{4} a^2)M$.

52. A hollow cylinder, with respect to its axis. *Ans.* $\frac{1}{2}(r_1^2 + r_2^2)M$.

53. A hollow sphere, with respect to a diameter. *Ans.* $\frac{8}{15}\pi(r_1^5 - r_2^5)$.

54. The sides of a rectangle are a, b. The perimeter being given, find the rectangle whose moment of inertia with respect to the side b is a maximum. *Ans.* $a = 3b$.

55. Solve Ex. 54 if the length of a diagonal, instead of the perimeter, is given. *Ans.* $a = \sqrt{3}\, b$.

56. Solve Ex. 55, using trigonometric functions (§ 40).

57. The total surface being given, find the circular cylinder whose moment of inertia with respect to the axis is a maximum. *Ans.* $a = \frac{3}{2} h$.

58. The slant height being given, find the circular cone whose moment of inertia with respect to the axis is a maximum. *Ans.* $a = 2h$.

59. Solve Ex. 58, using trigonometric functions.

60. The perimeter being given, find the isosceles triangle whose moment of inertia with respect to the base is a maximum. *Ans.* $h = \sqrt{15}\, a$.

135. Moment of inertia with respect to a plane. In most applications we are concerned with moment of inertia with respect to a line. Nevertheless it is frequently useful, as we shall see presently, to introduce the idea of moment of inertia with respect to a *plane*. The definitions and discussion of §§ 132–134 hold at once for the moment of inertia with respect to a plane if we replace the word "line" (or "axis") throughout by the word "plane."

Examples: (*a*) Find the moment of inertia of a sphere with respect to a diametral plane.

Taking the plane of moments as yz-plane and choosing as the element of volume a circular disk parallel to that plane, we have

$$I_{yz} = 2\pi \int_0^a x^2 y^2\, dx = 2\pi \int_0^a x^2(a^2 - x^2)\, dx = \tfrac{4}{15}\pi a^5 = \tfrac{1}{5} Ma^2.$$

(*b*) Find the moment of inertia of the solid in the example of § 120, with respect to the yz-plane.

Choosing the element as in Fig. 71, we have

$$I_{yz} = 2 \cdot \tfrac{1}{2} \int_0^2 x^2 yz\, dx = \int_0^2 x^2(4 - x^2)\, dx = \tfrac{64}{15} = \tfrac{4}{5} M.$$

EXERCISES

Find the following moments of inertia.

1. The following system of particles, with respect to each of the coördinate planes: 3 units at (0, 0, 2), 2 units at (4, 3, 2), 4 units at (− 2, 2, 1), 1 unit at (3, − 3, 0). *Ans.* $I_{yz} = 57$, $I_{zx} = 43$, $I_{xy} = 24$.

2. Equal particles at each corner of a cube, with respect to one of the faces. *Ans.* $\frac{1}{2} Ma^2$.

3. A rectangular parallelepiped, with respect to one face. *Ans.* $\frac{1}{3} Ma^2$.

4. A circular cylinder with respect to the base. Solve in two ways. *Ans.* $\frac{1}{3} Mh^2$.

5. A paraboloid of revolution bounded by a plane through the focus at right angles to the axis, (*a*) with respect to the plane tangent at the vertex; (*b*) with respect to the bounding plane.

Ans. (*a*) $\frac{1}{2} Ma^2$; (*b*) $\frac{1}{6} Ma^2$.

6. A circular cone with respect to the base. *Ans.* $\frac{1}{10} Mh^2$.

7. A circular cone with respect to the plane through the vertex parallel to the base.

8. An ellipsoid of revolution with respect to the plane through the center perpendicular to the axis. *Ans.* $\frac{1}{5} Ma^2$.

9. The volume of Ex. 46, p. 191, with respect to the yz-plane. *Ans.* $\frac{149}{245} Ma^2$.

10. A tetrahedron having three mutually perpendicular faces, with respect to one of those faces. *Ans.* $\frac{1}{10} Ma^2$.

11. An elliptic cone with respect to the base.

12. An ellipsoid with respect to each of the principal planes.

13. A right pyramid with a square base, with respect to the base. Solve in two ways. *Ans.* $\frac{1}{10} Mh^2$.

14. A right pyramid with a square base, with respect to a plane through the axis bisecting two sides of the base. *Ans.* $\frac{1}{6} Ma^2$.

15. The volume common to two equal cylinders of revolution whose axes intersect at right angles, with respect to the plane through the axes. (Cf. Ex. 12, p. 193.) *Ans.* $\frac{1}{6} Ma^2$.

16. The volume of Ex. 15, p. 193, with respect to the yz-plane. Solve in two ways.

17. Any cylindrical or prismatic solid bounded by two right sections, with respect to the base. *Ans.* $\frac{1}{3} Mh^2$.

18. Any cylindrical surface bounded by two right sections, with respect to one of the bounding planes. *Ans.* $\frac{1}{3} Mh^2$.

19. A sphere with respect to a diametral plane, by a new method. (Cf. Ex. 17.)

20. Ex. 6 by a new method.

21. Ex. 8 by a new method.

22. Example (*b*), § 135, by a new method.

23. The volume in example (*b*), § 135, with respect to the zx- and xy-planes. *Ans.* $I_{xx} = \frac{8}{5} M$, $I_{xy} = \frac{8}{15} M$.

24. The volume of Ex. 47, p. 191, with respect to the xy-plane. *Ans.* $\frac{1}{6} Ma^2$.

25. The volume of Ex. 49, p. 191, with respect to the xy-plane. *Ans.* $\frac{1}{5} Mc^2$.

26. A spherical surface with respect to a diametral plane, using (*a*) polar, (*b*) cartesian coördinates. *Ans.* $\frac{1}{3} Ma^2$.

27. A torus with respect to the plane through the center perpendicular to the axis. *Ans.* $\frac{1}{4} Ma^2$.

136. General theorems on moments of inertia. We proceed to state certain theorems by means of which the work of finding moments of inertia may in many cases be greatly simplified.

THEOREM I: *The moment of inertia of any mass with respect to a line is equal to the sum of the moments with respect to two perpendicular planes through the line.*

For example,

$$I_x = I_{xy} + I_{zx}.$$

COROLLARY: *The moment of inertia of a plane mass with respect to a line perpendicular to its plane is equal to the sum of the moments with respect to two lines in the plane intersecting at right angles in the foot of the perpendicular.*

For example, for a mass in the xy-plane,

$$I_z = I_x + I_y.$$

THEOREM II: *The moment of inertia of any mass with respect to a line (or plane) is equal to the moment with respect to the parallel centroidal line* * *(or plane) plus the product of the mass by the square of the distance between the lines (or planes).*

That is, if l is any line, $\bar{l}$ the parallel centroidal line, d the distance between them, then

$$I_l = I_{\bar{l}} + Md^2.$$

We shall prove these theorems only for a system of particles. The general proof is easily written out by means of multiple integrals.

To prove Theorem I, let us take the two perpendicular planes as the xy-plane and the zx-plane. Then, for a system of n particles,

$$I_x = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2),$$

$$I_{xy} = \sum_{i=1}^{n} m_i z_i^2,$$

$$I_{zx} = \sum_{i=1}^{n} m_i y_i^2.$$

* That is, the parallel line through the centroid.

Hence

$$I_x = I_{xy} + I_{zx}.$$

The proof of the corollary is left to the student.

To prove Theorem II for any line l and the parallel centroidal line $\bar{l}$, we note that in Fig. 85, by the cosine law,

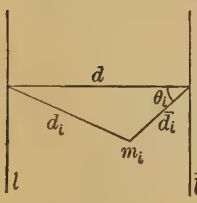


FIG. 85

$$\begin{aligned} d_i^2 &= \bar{d}_i^2 + d^2 - 2\,d\bar{d}_i \cos\theta_i \\ &= \bar{d}_i^2 + d^2 - 2\,dp_i, \end{aligned}$$

where p_i is the distance of m_i from the plane through $\bar{l}$ perpendicular to the plane of l and $\bar{l}$. Hence

$$\sum_{i=1}^{n} m_i d_i^2 = \sum_{i=1}^{n} m_i \bar{d}_i^2 + d^2 \sum_{i=1}^{n} m_i - 2\,d \sum_{i=1}^{n} m_i p_i.$$

But the quantity $\sum_{i=1}^{n} m_i p_i$ is the mass-moment of the system with respect to the plane through $\bar{l}$ perpendicular to the plane determined by l and $\bar{l}$, and since this perpendicular plane contains the centroid, the moment in question is 0. Hence

$$I_l = I_{\bar{l}} + Md^2.$$

The proof of Theorem II for two parallel planes is still simpler. It is left to the student.

EXERCISES

Find the following moments of inertia.

1. A circular wire with respect to (*a*) a tangent; (*b*) a line through the center perpendicular to the plane of the circle; (*c*) any line perpendicular to that plane. *Ans.* (*a*) $\frac{3}{2}Ma^2$; (*b*) Ma^2; (*c*) $M(a^2 + d^2)$.

2. A circular area with respect to (*a*) a tangent; (*b*) a perpendicular through the center; (*c*) a perpendicular through a point in the circumference. *Ans.* (*c*) $\frac{3}{2}Ma^2$.

3. A square plate with respect to (*a*) a line through the center parallel to a side; (*b*) a perpendicular through one corner; (*c*) a perpendicular through the center; (*d*) a line trisecting two opposite sides; (*e*) a diagonal.

Ans. (*a*) $\frac{1}{12}Ma^2$; (*b*) $\frac{2}{3}Ma^2$; (*c*) $\frac{1}{6}Ma^2$; (*d*) $\frac{1}{9}Ma^2$; (*e*) $\frac{1}{12}Ma^2$.

4. An isosceles triangle with respect to a line (*a*) parallel to the base bisecting the altitude; (*b*) through the vertex perpendicular to the plane.

Ans. (*a*) $\frac{1}{12} Mh^2$; (*b*) $\frac{1}{2} M(h^2 + \frac{1}{3} a^2)$.

5. The area in Fig. 76, p. 207, with respect to a line through the centroid (*a*) parallel to the base; (*b*) perpendicular to the plane.

Ans. (*a*) 60.2; (*b*) 94.5.

6. The area in Fig. 78, p. 207, with respect to a line through the centroid parallel to the base. *Ans.* 295.1.

7. The area in Fig. 80, p. 211, with respect to a perpendicular through the center of the circle. *Ans.* $80 + 4\pi$.

8. The area in Fig. 81, p. 211, with respect to a perpendicular through the center of the circle. *Ans.* $\frac{1280}{3} - 4\pi$.

9. A cube with respect to a line through the center parallel to an edge. *Ans.* $\frac{1}{6} Ma^2$.

10. A sphere with respect to a tangent.

11. An ellipsoid with respect to each of its axes.

12. A circular cylinder with respect to (*a*) a plane through the axis; (*b*) a generator; (*c*) a diameter of the middle cross section; (*d*) a line tangent to the base. *Ans.* (*d*) $(\frac{1}{3} h^2 + \frac{5}{4} a^2)M$.

13. A circular cone with respect to (*a*) a diameter of the base; (*b*) a line through the vertex perpendicular to the axis; (*c*) a diameter of the middle cross section.

Ans. (*b*) $(\frac{3}{5} h^2 + \frac{3}{20} a^2)\, M$; (*c*) $(\frac{1}{10} h^2 + \frac{3}{20} a^2)\, M$.

14. A right pyramid with a square base, with respect to a plane through the apex parallel to the base.

15. A torus with respect to a tangent plane parallel to the axis.

16. A cylindrical drum with respect to its axis.

137. Kinetic energy of a rotating body. The *kinetic energy* of a particle of mass m moving with a velocity v is defined as

$$E = \tfrac{1}{2} mv^2.$$

If a particle at a distance r from a fixed line rotates with an angular velocity ω about that line as an axis, its linear velocity is (§ 70)

$$v = \omega r;$$

hence its kinetic energy is

$$E = \tfrac{1}{2} m\omega^2 r^2.$$

The kinetic energy of a system of n particles rotating with angular velocity ω is

$$E = \tfrac{1}{2}\,\omega^2 \sum_{i=1}^{n} m_i r_i^2 = \tfrac{1}{2}\,I\omega^2,$$

where I is the moment of inertia of the system with respect to the axis of rotation. This formula holds in general.

A discussion of the various systems of units in actual use is outside the scope of this book. In the exercises below the so-called *engineering system* is used. In this system the mass m is replaced by the value

$$m = \frac{w}{g},$$

where w is the " weight " in pounds, and g the acceleration of gravity (32 ft. per sec. per sec. approximately). If then v is expressed in feet per second, and ω in radians per second, the energy is measured in " foot-pounds."

Example: A straight rod 4 ft. long, weighing 3 lbs. per ft., rotates about a perpendicular through one end at the rate of 60 R. P. M. (revolutions per minute). Find the kinetic energy.

By Ex. 6, p. 218, the moment of inertia is

$$I = \tfrac{1}{3}\,Ml^2 = \tfrac{1}{3} \cdot \tfrac{12}{32} \cdot 16\,;$$

the angular velocity is

$$\omega = 120\,\pi \text{ rad. per min.} = 2\,\pi \text{ rad. per sec.}$$

Hence

$$E = \tfrac{1}{2} \cdot \tfrac{1}{3} \cdot \tfrac{12}{32} \cdot 16 \cdot 4\,\pi^2 = 40 \text{ ft.-lbs., nearly.}$$

EXERCISES

1. A straight rod 10 ft. long, weighing 20 lbs., rotates about a perpendicular through one end at the rate of 20 R. P. M. Find its kinetic energy. *Ans.* 46 ft.-lbs.

2. A straight rod 6 ft. long, weighing 8 oz. per ft., rotates about a perpendicular through the midpoint at the rate of 30 R. P. M. Find its kinetic energy. *Ans.* 1.4 ft.-lbs.

3. A flywheel 12 ft. in diameter whose rim weighs 10 tons makes 50 R. P. M. Neglecting the mass of the spokes, find the kinetic energy of the wheel. *Ans.* 156 ft.-tons.

4. A flywheel 1 ft. in diameter, weighing 50 lbs., makes 100 R. P. M. If the wheel can be considered as a uniform circular disk, find its kinetic energy. *Ans.* 10.7 ft.-lbs.

5. A wheel 4 ft. in diameter has 8 spokes weighing 20 lbs. each. The rim weighs 600 lbs. Find the kinetic energy of the wheel when it is making 20 R. P. M. *Ans.* 180 ft.-lbs.

6. A thin spherical shell of radius 2 in., weighing 4 oz., rotates about a diameter at the rate of 2 R. P. S. Find the kinetic energy. *Ans.* 0.01 ft.-lb.

7. Solve Ex. 6 if the rotation takes place about an axis at a distance of 4 in. from the center. *Ans.* 0.08 ft.-lb.

8. A rectangular plate 4×3 ft., weighing 8 oz. per sq. ft., rotates about one of the shorter sides at the rate of 60 R. P. M. Find the kinetic energy. *Ans.* 20 ft.-lbs.

9. A cylindrical drum 6 ft. long and 2 ft. in diameter, weighing 4 oz. per sq. ft., revolves about the axis at the rate of 10 R. P. M. Find the kinetic energy. *Ans.* 0.175 ft.-lb.

10. The kinetic energy of a solid sphere 1 ft. in diameter making 60 R. P. M. about an axis through its center is 5 ft.-lbs. Find the weight of the sphere. *Ans.* 80 lbs.

11. A hollow cast-iron sphere (sp. gr. 7.5) 4 ft. in diameter rotates about a diameter at the rate of 1 rad. per sec. Its kinetic energy is 200 ft.-lbs. Find the inner radius. *Ans.* 1.7 ft.

I_d rod with to one end $\frac{1}{3} M a^2$

I_b rect to face $\frac{1}{6} M h^2$

I_o rect to diameter $\frac{1}{4} M a^2$

rect to center $\frac{1}{2} M a^2$

$1+3+5+7+\cdots\ 2n-1$

$S = n^2$

$2+4+6+\cdots\ 2n$

arithmetic series, $S_n = \frac{n(a+l)}{2} = \frac{n}{2}[a+(n-1)d]$

Geometric series, $S_n = \frac{a-ar^n}{1-r}$

CHAPTER XXI

SERIES OF CONSTANT TERMS

138. Series of n terms. A *series* of n terms is an expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n,$$

where each term is formed from the preceding one by some definite law. An example is the *geometric series*

$$a + ar + ar^2 + \cdots + ar^{n-1}.$$

139. Infinite series. When the number of terms increases indefinitely, a series of n terms becomes an *infinite series*, denoted by the symbol

$$u_1 + u_2 + u_3 + \cdots.$$

The series is defined by the *law of formation* of successive terms, or, what amounts to the same thing, by the *n-th or general term.*

The general term may frequently be written down by inspection of the first few terms, as in the following

Examples: (*a*) In the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

the general term is $\dfrac{1}{n}$.

(*b*) In the series

$$\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2\cdot 4}\cdot\frac{1}{5} + \frac{1}{2\cdot 4\cdot 6}\cdot\frac{1}{7} + \frac{1}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9} + \cdots,$$

the general term is * $\dfrac{1}{2^n\, n!}\cdot\dfrac{1}{2n+1}$.

* The symbol $n!$ (read factorial n) denotes the product of all the integers from 1 to n: $n! = 1\cdot 2\cdot 3\cdots n$.

140. Sum of an infinite series. It is shown in elementary algebra that the sum of a geometric series of n terms is

$$S_n = \frac{a - ar^n}{1 - r}. \tag{1}$$

Similarly the sum of any series of a finite number of terms can be found.

On the other hand, an infinite series has no sum in the ordinary sense of the term, since no matter how many terms we might add up, there would always be an infinite number left over. We may, however, give a meaning to the term " sum " even in this case by laying down the following *definition:*

The sum of an infinite series is defined as the limit, as n increases indefinitely, of the sum of the first n terms:

$$S = \lim_{n \to \infty} S_n,$$

provided the limit exists.

Thus the " sum " of an infinite series is the *limit* of an ordinary sum.

Example: By (1), the sum of the first n terms of the infinite geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

is

$$S_n = \frac{a - ar^n}{1 - r}.$$

Hence the sum of the series, if the sum exists, is

$$S = \lim_{n \to \infty} \frac{a - ar^n}{1 - r}.$$

When r is numerically less than 1, the quantity ar^n approaches 0 as n increases, and

$$S = \frac{a}{1 - r}.$$

When r is numerically greater than 1, the quantity ar^n increases indefinitely, and the above limit does not exist; the series has no sum.

141. Convergence and divergence. If the series has a sum S, *i.e.* if S_n approaches a limit when n increases, the series is said to be *convergent*, or to *converge to the value* S; if the limit does not exist, the series is *divergent*.

It follows from the above example that a geometric series converges to the value $\frac{a}{1-r}$ if $|r| < 1$; it diverges if $|r| > 1$.

A series may diverge, as in the case of a geometric series for which $r > 1$, because S_n increases indefinitely as n increases; or it may diverge because S_n increases and decreases alternately, or *oscillates*, without approaching any limit. In the latter case the series is called *oscillatory*.

In the elementary applications divergent series are of no importance. Before being able to use a given series we must determine whether it converges or diverges. If S_n can be expressed explicitly as a function of n, as in the case of the arithmetic and geometric series, we can in general determine the convergence or divergence of the series directly, and find the sum if it exists; but S_n cannot be so expressed in most cases.

A *necessary* condition for convergence is that the general term approach 0 as its limit:

$$\lim_{n \to \infty} u_n = 0.$$

For, when this condition is not satisfied, each term that is added changes S_n by an amount that does not approach 0, so that the difference between S_n and a fixed number S obviously cannot become and remain arbitrarily small.

This condition, though necessary, is not sufficient; *i.e.* if the condition is not satisfied, the series diverges, but if it is satisfied, the series still may diverge. This is illustrated by the *harmonic series*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots,$$

which will be shown in the next article to be divergent, although

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Many special tests for convergence have been devised, applicable to more or less broad classes of series. Several of the simplest are considered in the next few articles.

EXERCISES

Write down the general term of each of the following series.

1. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$.

2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$.

3. $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$.

4. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$.

5. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots$.

6. $1 + \frac{1!}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots$.

7. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$.

8. $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \cdots$.

9. $1 + \frac{1 \cdot 3}{2^2} + \frac{2 \cdot 4}{3^2} + \frac{3 \cdot 5}{4^2} + \cdots$.

10. $\frac{1}{2} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{5 \cdot 6 \cdot 7 \cdot 8} + \cdots$.

11. $\frac{1}{2} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5 \cdot 6} + \frac{4}{5 \cdot 6 \cdot 7 \cdot 8} + \cdots$.

12. Show that every infinite arithmetic series is divergent.

13. Find the sum of a geometric series of n terms for which $r = 1$, (*a*) directly, (*b*) by applying the theorem of § 72 to formula (1), § 140.

14. Show that the infinite geometric series for which $r = 1$ is divergent.

15. Show that the infinite geometric series for which $r = -1$, viz.

$$a - a + a - a + \cdots,$$

diverges by oscillation.

16. The second term of an infinite geometric series being given, what value of r gives the smallest sum? *Ans.* $\frac{1}{2}$.

17. The third term of an infinite geometric series being given, what value of r gives the smallest sum? *Ans.* $\frac{2}{3}$.

18. Show that the convergence or divergence of a series is not affected by dropping any finite number of terms from the series.

Prove that the following series are divergent.

19. $1 + 2 + 4 + 8 + \cdots$.

20. $1 - 2 + 3 - 4 + \cdots$.

21. $1 + \frac{1}{2} - 1 - \frac{1}{2} + 1 + \frac{1}{4} - 1 - \frac{1}{5} + \cdots$.

22. $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \cdots$.

23. $1 - \frac{5}{8} + \frac{10}{18} - \frac{17}{32} + \frac{26}{50} - \cdots$.

24. $1 + \frac{1}{2} + \frac{1}{4} + 1 + \frac{1}{8} + \frac{1}{16} + 1 + \frac{1}{32} + \cdots$.

142. Integral test. Consider the following

Example: Prove that the harmonic series

(1) $$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

is divergent.

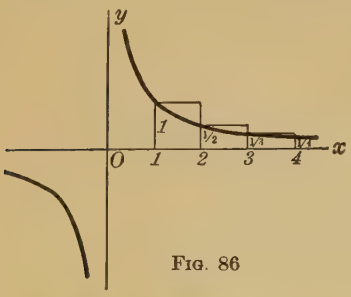


FIG. 86

Here the general term is

$$u_n = f(n) = \frac{1}{n}.$$

Let us draw the curve

$$y = f(x) = \frac{1}{x},$$

erect the ordinates at $x = 1, 2, 3, \cdots, n$, and complete the circumscribed rectangles as shown in the figure. Then the areas of the rectangles are, respectively, $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$, so that the sum of these areas is the sum S_n of the first n terms of the series (1). But the sum of the rectangles is clearly greater than the area under the curve from $x = 1$ to $x = n$:

$$S_n > \int_1^n y\,dx = \int_1^n \frac{dx}{x} = \log n.$$

When n increases, the area $\log n$ under the curve becomes infinite, hence S_n does likewise and the series diverges.

This example illustrates the *integral test:*

Given a series of *positive terms*

(2) $$u_1 + u_2 + u_3 + \cdots,$$

put

$$u_n = f(n).$$

If the function $f(x)$ is defined not only for positive integral values, but for all positive values of x, and if $f(x)$ never increases with x, then the series (2) *converges or diverges according as the integral* $\int_1^\infty f(x)\,dx$ *does or does not exist.*

The proof of this test is easily written out by drawing the curve $y = f(x)$ and following the process suggested by the above example; the details are left to the student. Of course in applying the test it is unnecessary to do more than evaluate the improper integral $\int_1^\infty f(x)\,dx$, or, what amounts to the same thing, $\int_1^\infty u_n\,dn$.

EXERCISES

Test the following series for convergence.

1. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$. *Ans.* Divergent.
2. $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \cdots$. *Ans.* Divergent.
3. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$. *Ans.* Convergent.
4. $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$. *Ans.* Convergent.
5. $1 + \frac{1}{2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \frac{1}{1+4^2} + \cdots$.
6. $\frac{2}{1+2^2} + \frac{3}{1+3^2} + \frac{4}{1+4^2} + \cdots$.
7. $\frac{1}{1+k} + \frac{1}{2+k} + \frac{1}{3+k} + \cdots$.
8. $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$. *Ans.* Conv. if $p > 1$, div. if $p \leqq 1$.
9. $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots$.
10. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.
11. $\frac{1}{e} + \frac{2^2}{e^2} + \frac{3^2}{e^3} + \frac{4^2}{e^4} + \cdots$.
12. $\frac{1}{2} + \frac{1}{2+2^3} + \frac{1}{3+3^3} + \frac{1}{4+4^3} + \cdots$.
13. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.
14. Write out the proof of the integral test.
15. In the statement of the integral test, why is it assumed that $f(x)$ never increases with x? Show that it would be sufficient to assume that $f(x)$ never increases with x after some fixed point $x = x_1$.

143. Comparison test. Let

$$u_1 + u_2 + u_3 + \cdots$$

be a series of *positive terms* to be tested.

(*a*) *If a series*

$$a_1 + a_2 + a_3 + \cdots$$

of positive terms, known to be convergent, can be found such that

$$u_n \leqq a_n,$$

then the series to be tested is convergent.

(*b*) *If a series*

$$b_1 + b_2 + b_3 + \cdots$$

of positive terms, known to be divergent, can be found such that

$$u_n \geqq b_n,$$

then the series to be tested is divergent.

To establish this test, we will need the following theorem on limits, which may be considered as self-evident.

THEOREM: *If a variable steadily* $\left\{\begin{matrix} increases \\ decreases \end{matrix}\right\}$ *but never becomes* $\left\{\begin{matrix} greater \\ less \end{matrix}\right\}$ *than some fixed number* A, *the variable approaches a limit which is not* $\left\{\begin{matrix} greater \\ less \end{matrix}\right\}$ *than* A.

Now, to prove part (*a*) of the test, let $S_n(u)$ be the sum of the first n terms of the u-series, $S_n(a)$ the sum of the first n terms of the a-series, and $S(a)$ the sum of the a-series. Since all the terms u_n are positive, $S_n(u)$ always increases with n. On the other hand, we have

$$S_n(u) < S_n(a) < S(a).$$

Since $S_n(u)$ always increases, but never exceeds the fixed number $S(a)$, it approaches a limit, by the above theorem, which is not greater than $S(a)$.

The proof of (*b*) is left to the student.

The success of the test depends on our ability to find a *convergent* series whose terms are *greater* than the corresponding terms of the series to be tested, or a *divergent* series whose

terms are *less* than those of the series to be tested. To show that the terms of the u-series are greater than those of some convergent series, or less than those of some divergent series, proves nothing.

Since the convergence of a series is not affected by discarding any finite number of terms, it is clear that the conditions of the test do not need to be satisfied from the very beginning of the series, but only *after a certain point*, all the terms up to that point being neglected.

Example: Test the series

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots. \tag{1}$$

The series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is known to converge (Ex. 8, p. 233). Discarding the first term of the series to be tested, we have

$$u_n = \frac{1}{n(n+1)} = \frac{1}{n^2 + n}.$$

The general term of the known series is

$$a_n = \frac{1}{n^2}.$$

Since

$$\frac{1}{n^2 + n} < \frac{1}{n^2}$$

for all positive values of n, the series (1) converges.

EXERCISES

Test the following series for convergence by the comparison test.

1. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$
2. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots.$
3. $\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots.$
4. $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots.$
5. $e + \frac{1}{2}e^2 + \frac{1}{3}e^3 + \cdots.$

6. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots$.

7. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{4\cdot 5\cdot 6}+\frac{1}{7\cdot 8\cdot 9}+\cdots$.

8. $\frac{1}{4}+\frac{1}{8}+\frac{1}{14}+\frac{1}{19}+\cdots$.

9. $\frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^9}+\frac{1}{2^{16}}+\cdots$.

10. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots$.

11. $\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^{10}}+\cdots$.

12. $\frac{\sqrt{2}}{1}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\cdots$.

144. Ratio test. There are many "ratio tests"; the simplest is the following:

Given the series

$$u_1 + u_2 + u_3 + \cdots,$$

form the ratio $\frac{u_{n+1}}{u_n}$ of a general term* to the one preceding it.

(*a*) *If* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$, *the series converges.*

(*b*) *If* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$, *or if* $\left|\frac{u_{n+1}}{u_n}\right|$ *increases indefinitely, the series diverges.*

(*c*) *If* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$, *the test fails.*

This test holds for any series whatever, not merely for series of positive terms.

Suppose we have case (*a*): $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L < 1$. At present we shall consider only the case in which all the terms are positive, and suggest in § 146 a method of completing the proof. Let us choose some number r between L and 1. By the definition of limit, the difference between the ratio $\frac{u_{n+1}}{u_n}$ and its limit L ultimately becomes

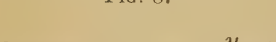


FIG. 87

* We may divide the $(n+1)$-th term by the n-th, the n-th by the $(n-1)$-th — *any* general term by the one before it, since the question of convergence is not affected by dropping any finite number of terms.

and remains as small as we please; therefore a number m can be found such that for all values of $n \geqq m$ we have

$$\frac{u_{n+1}}{u_n} < r.$$

Hence

$$\begin{aligned} u_{m+1} &< u_m r, \\ u_{m+2} &< u_{m+1} r < u_m r^2, \\ u_{m+3} &< u_{m+2} r < u_m r^3, \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \ . \end{aligned}$$

Discarding the first m terms of our series, we see that the remaining terms are less than the corresponding terms of the series

$$u_m r + u_m r^2 + u_m r^3 + \cdots.$$

But this latter series, being a geometric series with ratio $r < 1$, is convergent; hence the given series is convergent, by the comparison test.

The above argument is readily modified to prove case (b). The details are left to the student.

The test may be shown to fail in case (c) by the following

Example: For the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

the test ratio is

$$\frac{\dfrac{1}{(n+1)^p}}{\dfrac{1}{n^p}} = \left[\frac{n}{n+1}\right]^p = \left[\frac{1}{1+\dfrac{1}{n}}\right]^p,$$

and

$$\lim_{n \to \infty} \left[\frac{1}{1+\dfrac{1}{n}}\right]^p = 1.$$

But, by Ex. 8, p. 233, when $p \leqq 1$, this series diverges; when $p > 1$, the series converges. Hence there are both convergent and divergent series for which the limit of the test ratio is 1.

145. Alternating series. A series whose terms are alternately positive and negative is called an *alternating series.* Such series are of frequent occurrence in the elementary applications. Their most important properties are embodied in the following theorems.

THEOREM I: *If after a certain point the terms of an alternating series never increase numerically, and if the limit of the n-th term is* 0, *the series is convergent.*

THEOREM II: *In an alternating series that converges by Theorem* I, *the difference between the sum of the series and the sum of the first n terms is not greater numerically than the* $(n + 1)$*-th term:*

$$|S - S_n| \leqq |u_{n+1}|.$$

To prove Theorem I, let us write the series in the form

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

where all the u's are positive. When n is even, say $n = 2m$, we may write S_n in the two forms

(1) $S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}),$

(2) $S_{2m} = u_1 - (u_2 - u_3) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}.$

Equation (1) shows that S_{2m} always increases, since each of the parentheses is positive. Equation (2) shows that S_{2m} is always less than u_1. Since S_{2m} always increases, but never exceeds the fixed number u_1, it approaches a limit S not greater than u_1, by the theorem on limits stated in § 143. Further, the sum of an odd number of terms S_{2m+1} approaches the same limit, since

$$\lim_{n \to \infty} (S_{2m+1} - S_{2m}) = \lim_{n \to \infty} u_{2m+1} = 0.$$

Theorem II follows at once. For, if n is even, the difference $S - S_n$ is the alternating series

$$u_{n+1} - u_{n+2} + u_{n+3} - \cdots,$$

and it follows from (2) that the sum of this series is not greater than the first term. Similarly if n is odd.

146. Absolute convergence. A series is said to be *absolutely convergent* if the series formed from it by replacing all its terms by their absolute values is convergent. The idea of absolute convergence derives its importance from the fact that absolutely convergent series possess certain important properties not belonging to other series (*e.g.* see § 153).

It is easily shown that a series always converges if the series of absolute values converges. From this fact the proof in § 144 is easily completed for the case when the terms are not all positive.

EXERCISES

Determine whether the following series are convergent or divergent.

1. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots.$
2. $1-\frac{1}{2!}+\frac{1}{3!}-\cdots.$
3. $\frac{e^2}{2}+\frac{e^3}{3}+\frac{e^4}{4}+\cdots.$
4. $\frac{\pi}{4}+2\left(\frac{\pi}{4}\right)^2+3\left(\frac{\pi}{4}\right)^3+\cdots.$
5. $\frac{1!}{10}+\frac{2!}{10^2}+\frac{3!}{10^3}+\cdots.$
6. $\frac{2}{1!}-\frac{3}{2!}+\frac{4}{3!}-\cdots.$
7. $\frac{1}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\cdots.$
8. $\frac{1}{e}+\frac{2^{10}}{e^2}+\frac{3^{10}}{e^3}+\cdots.$
9. $1-\frac{1}{3!}+\frac{1}{5!}-\cdots.$
10. $\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}+\cdots.$
11. $\frac{1}{2^2}\cdot\frac{1}{2^2}+\frac{1}{3^2}\cdot\frac{1}{2^3}+\frac{1}{4^2}\cdot\frac{1}{2^4}+\cdots.$
12. $\frac{1}{2}+\frac{2^2}{1+2^3}+\frac{3^2}{1+3^3}+\cdots.$
13. $2+\frac{2\cdot 2^2}{1!}+\frac{3\cdot 2^3}{2!}+\frac{4\cdot 2^4}{3!}+\cdots.$
14. $\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2\cdot 4}\cdot\frac{1}{5}+\frac{1}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\frac{1}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9}+\cdots.$
15. $\frac{1}{2}\cdot\frac{1}{3}+\frac{2}{2\cdot 4}\cdot\frac{1}{5}+\frac{3}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\frac{4}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9}+\cdots.$
16. $\frac{1}{3}+\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\cdots.$
17. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$
18. $1-\frac{1}{3^2}+\frac{1}{5^2}-\cdots.$

19. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. **20.** $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$.

21. Which of the series in Exs. 2, 6, 9, 17, 18, 19, 20 are absolutely convergent?

22. Find the largest term of the series in Ex. 8. *Ans.* $n = 10$.

23. Carry out the proof of case (*b*), § 144.

24. If $a_1 + a_2 + a_3 + \cdots$ is an absolutely convergent series and $b_1, b_2, b_3, \cdots$ a set of numbers that remain finite as n increases: $b_n < M$, where M is a constant, show that the series

$$a_1b_1 + a_2b_2 + a_3b_3 + \cdots$$

converges absolutely.

25. If $a_1 + a_2 + a_3 + \cdots$ is an absolutely convergent series and if $u_1 + u_2 + u_3 + \cdots$ is a series such that $\frac{u_n}{a_n}$ approaches a limit when n increases, show that the u-series converges absolutely.

26. State and prove a theorem for divergent series analogous to that of Ex. 25.

147. Evaluation of the sum of a series. So far we have striven merely to determine whether a given series is convergent or divergent — *i.e.* whether it does or does not have a sum. The existence of a sum having been established, the next problem is to determine its value. It follows from the definition of convergence that this can be done to any desired degree of approximation by merely adding together a sufficient number of terms at the beginning of the series*. However, unless the series is "rapidly convergent" — *i.e.* unless the terms diminish rapidly in numerical value — the amount of computation involved in this process is apt to be prohibitive†.

148. Limit of error. In computing the sum of a series by addition of terms, it is necessary to know an upper limit for the error committed by stopping with any given term.

* In some cases, as noted in § 141, the sum can be found exactly, but this is not usually the case.

† More elaborate methods, beyond the range of the present discussion, make it possible to sum many slowly convergent series with comparatively little labor.

In the case of alternating series this question is answered at once by Theorem II of § 145:

The error committed by stopping at any point of a convergent alternating series * is *less than the first term neglected.*

If a series converges rapidly, it is usually easy to show that, even though the succeeding terms all have the same sign, the error committed by stopping at any point is only slightly greater than the first term neglected (cf. the example below).

As a rule, the terms retained in the computation are replaced by decimal approximations, and care must be taken to see that the errors thus introduced do not accumulate sufficiently to affect the result. If the sum is to be correct to k decimal places, each term must be computed to at least $k + 1$ places, and frequently more.

Example: Find the sum of the series

$$\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \cdots$$

correct to four decimal places.

This series is easily proved convergent by the ratio test. To find the sum, we have

$$\frac{1}{5} = 0.2, \quad \frac{1}{3 \cdot 5^3} = 0.00267, \quad \frac{1}{5 \cdot 5^5} = 0.00006.$$

The fourth term in itself is evidently much too small to affect the fifth place, and the error committed by stopping with the third term is but slightly greater than the fourth term†. Adding the three terms computed above, and discarding the fifth place as untrustworthy, we find the sum to be 0.2027.

* Of the type considered in § 145.

† The argument is as follows:

$$\frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \frac{1}{11 \cdot 5^{11}} + \cdots < \frac{1}{7 \cdot 5^7} + \frac{1}{7 \cdot 5^9} + \frac{1}{7 \cdot 5^{11}} + \cdots$$

$$< \frac{1}{7 \cdot 5^7}\left(1 + \frac{1}{5^2} + \frac{1}{5^4} + \cdots\right).$$

The series in parenthesis is a geometric series whose sum is $\dfrac{1}{1 - \dfrac{1}{25}} = \dfrac{25}{24}$.

Thus the error in stopping with the third term is less than $\frac{25}{24}$ times the fourth term.

EXERCISES

Compute the sums of the following series, correct to the number of decimal places indicated in the respective answers.

1. $1 - \frac{1}{10} + \frac{2}{10^2} - \frac{3}{10^3} + \cdots$. *Ans.* 0.91736.

2. $1 - \frac{1}{3!} + \frac{1}{5!} - \cdots$. *Ans.* 0.84147.

3. $1 - \frac{1}{2!} + \frac{2}{3!} - \frac{3}{4!} + \cdots$. *Ans.* 0.736.

4. $\frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \cdots$. *Ans.* 0.04.

5. $\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$. *Ans.* 0.69.

6. $1 + \frac{1}{2} + \frac{1}{2^2 \cdot 2!} + \frac{1}{2^3 \cdot 3!} + \cdots$. *Ans.* 1.6487.

7. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \cdot \frac{1}{10} + \frac{1}{3 \cdot 4} \cdot \frac{1}{10^2} + \frac{1}{4 \cdot 5} \cdot \frac{1}{10^3} + \cdots$. *Ans.* 0.5176.

8. $\frac{1}{2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{3^2 \cdot 2^3} + \frac{1}{4^2 \cdot 2^4} + \cdots$. *Ans.* 0.58.

CHAPTER XXII

POWER SERIES

149. Power series. In the previous chapter we considered only series whose terms are constants. The case of greatest practical importance, however, is that in which the terms are functions of a variable.

A series of the form

$$a_0 + a_1x + a_2x^2 + \cdots,$$

where x is a variable and a_0, a_1, a_2, $\cdots$ are constants, is called a *power series*. Such series are of especial importance in practice, and will form the subject of this chapter.

A power series may converge for all values of the variable x, or for no values except 0; but usually it will converge for all values in some finite interval, and diverge for all values outside that interval. The interval of convergence always extends equal distances on each side of the point $x = 0$.

The interval of convergence can usually be determined by the ratio test. We illustrate the process by an

Example: Find the interval of convergence of the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

Here

$$u_n = \frac{x^n}{n},$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot|x| = |x|,$$

by the theorem * of § 72. Therefore:

(*a*) The series converges when $|x| < 1$, *i.e.* $-1 < x < 1$.

(*b*) The series diverges when $|x| > 1$.

(*c*) The test fails when $x = \pm 1$. But when $x = 1$, the series is the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

and therefore diverges; when $x = -1$, the series is

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$

which converges by § 145.

Hence the interval of convergence is $-1 \leqq x < 1$.

EXERCISES

Find the interval of convergence of the following series.

1. $1 + 2x + 3x^2 + 4x^3 + \cdots$. *Ans.* $-1 < x < 1$.

2. $1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \cdots$. *Ans.* $-1 \leqq x < 1$.

3. $1 - x + \frac{x^2}{3} - \frac{x^3}{5} + \cdots$. *Ans.* $-1 < x \leqq 1$.

4. $1 - \frac{2x}{3} + \frac{3x^2}{9} - \frac{4x^3}{27} + \cdots$. *Ans.* $-3 < x < 3$.

5. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. *Ans.* All values of x.

6. $1 + 10x + 2 \cdot 100x^2 + 3 \cdot 1000x^3 + \cdots$.

7. $1 - 3x + 5x^2 - 7x^3 + \cdots$.

8. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. *Ans.* All values of x.

9. $1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$. *Ans.* $x = 0$.

10. $1 + x + 4x^2 + 9x^3 + 16x^4 + \cdots$.

11. $1 + \frac{x}{2} + \frac{2^2x^2}{2^2} + \frac{3^2x^3}{2^3} + \frac{4^2x^4}{2^4} + \cdots$. *Ans.* $-2 < x < 2$.

* Objection may be raised to the use of this theorem in the present instance, on the ground that n is not a continuous variable. The objection, however, is easily removed. For, if we can prove by the theorem that a given function of n approaches a certain limit when n varies continuously, it is certain that the same limit will be approached when n varies through positive integral values.

12. $x + \frac{x^2}{1+2^2} + \frac{x^3}{1+3^2} + \frac{x^4}{1+4^2} + \cdots$. *Ans.* $-1 \leqq x \leqq 1$.

13. $x + \frac{x^2}{1+2^2} + \frac{x^3}{2+3^2} + \frac{x^4}{3+4^2} + \cdots$.

14. $\frac{1}{1\cdot 2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \cdots$.

15. $\frac{1}{1\cdot 3} + \frac{x}{3\cdot 5} + \frac{x^2}{5\cdot 7} + \frac{x^3}{7\cdot 9} + \cdots$.

16. $\frac{x^3}{3^2} + \frac{2}{3}\cdot\frac{x^5}{5^2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^7}{7^2} + \cdots$. *Ans.* $-1 \leqq x \leqq 1$.

17. $1 + \frac{x^2}{2} + \frac{3}{2}\frac{x^4}{4} + \frac{3\cdot 5}{2\cdot 4}\frac{x^6}{6} + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\frac{x^8}{8} + \cdots$. *Ans.* $-1 < x < 1$.

18. $1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots$.
Ans. $-1 < x < 1$.

19. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$. *Ans.* $0 < x \leqq 2$.

20. $(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{9}(x-1)^3 - \cdots$. *Ans.* $0 \leqq x \leqq 2$.

21. $(x-3) + 2(x-3)^2 + 3(x-3)^3 + \cdots$. *Ans.* $2 < x < 4$.

22. $1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots$. *Ans.* All values of x.

23. $1 + (x+1) + \frac{(x+1)^2}{3} + \frac{(x+1)^3}{5} + \cdots$. *Ans.* $-2 \leqq x < 0$.

24. $1 + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \cdots$. *Ans.* $x \leqq -1$, and $x > 1$.

25. $1 - \frac{2^2}{x} + \frac{3^2}{x^2} - \frac{4^2}{x^3} + \cdots$. *Ans.* $x < -1$, and $x > 1$.

26. $\frac{1}{x} + \frac{1\cdot 3}{x^2} + \frac{1\cdot 3\cdot 5}{x^3} + \cdots$. *Ans.* Divergent for all values of x.

27. $1 + \frac{1}{2x} + \frac{2}{4x^2} + \frac{3}{8x^3} + \cdots$. *Ans.* $x < -\frac{1}{2}$, and $x > \frac{1}{2}$.

28. Find the sum of the series in Ex. 1, (*a*) for $x = \frac{1}{10}$; (*b*) for $x = \frac{1}{5}$.
Ans. (*a*) 1.235; (*b*) 1.56.

29. Find the sum of the series in Ex. 2 for $x = -\frac{1}{3}$. *Ans.* 0.940.

30. Find the sum of the series in Ex. 11 to four decimal places, (*a*) for $x = -\frac{1}{10}$; (*b*) for $x = \frac{1}{10}$. *Ans.* (*a*) 0.9590; (*b*) 1.0612.

31. Find the sum of the series in Ex. 20 for $x = 0.99$.
Ans. -0.01003.

32. Find the sum of the series in Ex. 22 for $x = 2.2$. *Ans.* 1.2214.

33. Find the sum of the series in Ex. 24 for $x = 10$. *Ans.* 1.1054.

150. Maclaurin's series. Being given a power series in x, if we substitute for x any value within the interval of convergence, the sum of the series is determined; *i.e.* the sum is a function of x. This suggests a very important problem: Being given any function of x, we may think of this function as the *sum of a certain power series*, and try to determine that series.

To do this, let us assume that such a series exists, and write

$$(1) \qquad f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots,$$

where the coefficients c_0, c_1, c_2, $\cdots$ are constants to be determined. Letting $x = 0$, we get $f(0) = c_0$; *i.e.* c_0 is the value of the given function at $x = 0$. Differentiating (1),

$$f'(x) = c_1 + 2\,c_2x + 3\,c_3x^2 + \cdots,$$

and setting $x = 0$, we find

$$f'(0) = c_1.$$

Proceeding in this way, we find

$$\begin{aligned} f''(0) &= 2 \cdot 1\, c_2, \\ f'''(0) &= 3 \cdot 2 \cdot 1\, c_3, \\ &\vdots \qquad\quad \vdots \\ f^{(n)}(0) &= n\,!\, c_n, \\ &\vdots \qquad\quad \vdots \end{aligned}$$

Hence (1) takes the following form, called *Maclaurin's series:*

$$(2) \qquad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

It should be noted that we have not proved the validity of this result; we have merely proved that if there is a series of the form (1) whose sum is $f(x)$, that series is given by equation (2). Evidently the series (2) can always be *formally* written down whenever the function and its successive derivatives are defined at $x = 0$, but cases can be found in which the sum of the series is not the given function. However, it is proved in more advanced texts that for the elementary functions of practical importance the series, if it converges for a given value of x, must converge to the value

$f(x)$; hence for all functions that we shall consider, *the interval within which Maclaurin's series is valid coincides with the interval of convergence of the series.* Within that interval the series is said to *represent the function,* and the function is said to be *developed* or *expanded* in powers of x.

One expansion, with which the student is already familiar, is the binomial theorem

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots,$$

this being valid for x numerically less than 1. Other expansions will now be derived.

Examples: (*a*) Expand $\sin x$ in powers of x.

Here

$$\begin{array}{lll} f(x) = \sin x, & \text{hence} & f(0) = 0, \\ f'(x) = \cos x, & & f'(0) = 1, \\ f''(x) = -\sin x, & & f''(0) = 0, \\ f'''(x) = -\cos x, & & f'''(0) = -1, \\ f^{(4)}(x) = \sin x, & & f^{(4)}(0) = 0, \\ f^{(5)}(x) = \cos x, & & f^{(5)}(0) = 1, \\ \vdots \quad \vdots & & \vdots \quad \vdots \end{array}.$$

Substituting in (2) the values found in the right-hand column, we find

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

It was shown in Ex. 8, p. 244, that this series converges for all values of x.

(*b*) Develop e^x in Maclaurin's series.

Here

$$\begin{array}{lll} f(x) = e^x, & \text{hence} & f(0) = 1, \\ f'(x) = e^x, & & f'(0) = 1, \\ f''(x) = e^x, & & f''(0) = 1, \\ \vdots \quad \vdots & & \vdots \quad \vdots \end{array}.$$

Therefore

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

By Ex. 5, p. 244, this series converges for all values of x.

151. Applications of Maclaurin's series. The most elementary application of series, and one of great importance, is in computing tables of values of various functions, *e.g.* logarithms, trigonometric functions, etc.

Examples: (*a*) Compute sin 3° correct to five decimal places.

Setting $x = 3° = \frac{\pi}{60}$ in the series for sin x obtained above, we get

$$\sin 3° = \sin \frac{\pi}{60} = \frac{\pi}{60} - \frac{1}{6}\left(\frac{\pi}{60}\right)^3 + \frac{1}{120}\left(\frac{\pi}{60}\right)^5 - \cdots$$
$$= 0.052360 - 0.000024 + \cdots.$$

Since this is an alternating series, the error committed by stopping with any term is less than the next term. Without computing the third term, we see that it is much too small to affect the sixth decimal place, hence we need keep only two terms:

$$\sin 3° = 0.05234.$$

(*b*) Since

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

it follows that for small values of the angle, sin θ and θ are nearly equal. Within what interval can the sine be replaced by the angle if the allowable error is 0.0005?

From the equation

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$

it appears that the error committed by stopping with the first term, *i.e.* by setting

$$\sin \theta = \theta,$$

is less than $\frac{1}{6}\theta^3$. We therefore have

$$\tfrac{1}{6}\theta^3 = 0.0005,$$
$$\theta^3 = 0.003,$$
$$\theta = 0.1442 \text{ (radian)}$$
$$= 8° 15', \text{ approximately.}$$

That is, the sine of any angle less than 8° 15′ can be replaced by the angle with an error less than 0.0005.

152. The value of e. As a further application of Maclaurin's series, let us derive the value of e given in § 50. Taking $x = 1$ in example (b), § 150, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

This gives *

$$\begin{array}{r} 1. \\ 1. \\ 0.5 \\ 0.166667 \\ 0.041667 \\ 0.008333 \\ 0.001389 \\ 0.000198 \\ 0.000025 \\ \underline{0.000003} \\ 2.71828. \end{array}$$

It is easily shown that the error is but little greater than the first term neglected.

EXERCISES

In Exs. 1–9, determine the interval of convergence of the series.

1. Expand $\cos x$ in powers of x.

Ans. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$, all values of x.

2. Expand $\sin kx$ in powers of x, (a) directly, (b) by substituting kx for x in example (a), § 150.

3. Expand $\cos kx$ in powers of x.

4. Develop $\frac{1}{1+x}$ in powers of x.

Ans. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad -1 < x < 1.$

5. Develop $e^{-\frac{1}{2}x^2}$ in powers of x, (a) directly, (b) by substitution in example (b), § 150.

6. Expand $\log(1 + x)$ in Maclaurin's series.

Ans. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x \leqq 1.$

* Note that the fourth term can be obtained by dividing the third by 3, the fifth by dividing the fourth by 4, etc.

7. Expand $\log(1-x)$ in Maclaurin's series, (*a*) directly, (*b*) by replacing x by $-x$ in Ex. 6.

$$Ans.\ \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad -1 \leqq x < 1.$$

8. Develop the function $\dfrac{x}{(1-x)^2}$ in powers of x.

$$Ans.\ \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots, \quad -1 < x < 1.$$

9. Obtain the *binomial theorem*

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots,$$

valid for $-1 < x < 1$.

10. Show that $\cot x$ cannot be expanded in powers of x.

11. Show that $x \cot x$ cannot be expanded in powers of x.

12. Show that $\log x$ cannot be expanded in powers of x.

13. Show that $x^n \log x$ cannot be expanded in powers of x.

14. Compute (*a*) $\sin 30'$ to six places, (*b*) $\sin 1°$ to five places, (*c*) $\sin 9°$ to three places.

15. Compute (*a*) $\cos 20'$, (*b*) $\cos 3°$, (*c*) $\cos 5°$, each to four places.

16. Find the value of e to ten decimal places.

$$Ans.\ e = 2.7182818285.$$

17. Find the tenth root of e. *Ans.* 1.10517.

18. Compute $\dfrac{1}{e}$ by series. *Ans.* 0.36788.

19. Extract the cube root of e. *Ans.* 1.396.

20. By means of Ex. 6, find the sum of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots. \qquad Ans.\ \log 2.$$

21. By means of Ex. 7, find the sum of the series

$$\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots.$$

(Cf. Ex. 5, p. 242.) *Ans.* $\log 2$.

22. By means of Ex. 7, find the sum of the series

$$\tfrac{2}{3} + \tfrac{1}{2}(\tfrac{2}{3})^2 + \tfrac{1}{3}(\tfrac{2}{3})^3 + \cdots. \qquad Ans.\ \log 3.$$

23. Using Ex. 8, sum the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots.$$

24. Taking $x = -\tfrac{1}{2}$ in Ex. 8, sum the series

$$\frac{1}{2} - \frac{2}{2^2} + \frac{3}{2^3} - \frac{4}{2^4} + \cdots.$$

25. Within what interval can $\sin\theta$ be replaced by θ, if the allowable error is 0.0001? *Ans.* $\theta < 4° 50'$.

26. Within what interval can $\sin\theta$ be replaced by $\theta - \frac{1}{6}\theta^3$, if the allowable error is 0.0001? *Ans.* $\theta < 23° 39'$.

27. Within what interval can $\cos\theta$ be replaced by $1 - \frac{1}{2}\theta^2$, if the allowable error is 0.0005? *Ans.* $\theta < 18° 58'$.

28. Within what interval does $\cos\theta$ differ from unity by less than 0.00005? *Ans.* $\theta < 34'$.

29. Compute log 1.06. *Ans.* 0.058.

30. Compute log 0.9. *Ans.* -0.105.

31. Raise 1.02 to the tenth power by use of the binomial theorem. *Ans.* 1.219.

32. Raise 0.9 to the eighth power. *Ans.* 0.4305.

33. Extract the square root of 101 by writing

$$(101)^{\frac{1}{2}} = (100 + 1)^{\frac{1}{2}} = 10(1 + \tfrac{1}{100})^{\frac{1}{2}}.$$ *Ans.* 10.0499.

34. Extract the fourth root of 15. *Ans.* 1.968.

35. Extract the cube root of 9. *Ans.* 2.08.

36. Solve the equation $\cos x = 10\,x$. *Ans.* $x = 0.0995$.

37 Find the points of intersection of the curves $y = e^x$, $y = \frac{9}{10}x + 1$. *Ans.* (0, 1), (− 0.215, 0.806).

38. Find the amount by which an arc of a great circle of the earth 1 mile long recedes from its chord. (Take $R = 4000$ mi.) *Ans.* 2 in.

39. Solve Ex. 38 if the arc is $2\frac{1}{2}$ miles long. *Ans.* 1 ft.

40. In Ex. 38, how much longer is the arc than the chord?

41. If a straight tunnel were to be bored through the earth from Detroit to Chicago (say 300 mi.), how much distance would be saved? *Ans.* 371 ft.

42. In Ex. 41, what would be the greatest depth of the tunnel? *Ans.* 2.8 mi.

43. Taking the earth's circumference as 40,000,000 meters, find the difference between the circumference and the perimeter of a regular inscribed polygon of 1,000,000 sides. *Ans.* Less than $\frac{1}{15}$ mm.

44. In § 152, the terms neglected are

$$\frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \cdots.$$

By comparison with the geometric series

$$\frac{1}{10!}\left(1 + \frac{1}{11} + \frac{1}{11^2} + \cdots\right),$$

find an upper limit for the error.

153. Operations with power series. Operations that can always be performed upon series of a finite number of terms, such as rearrangement of terms, multiplication of one series by another, term-by-term differentiation or integration, etc., cannot be assumed offhand to be allowable with infinite series, and in fact it is easily shown that they are not allowable in all cases.

In dealing with power series, it is frequently desirable to know just when such elementary operations are permissible. We therefore state the following theorems regarding power series; the proofs belong to a more advanced treatment of the subject.

THEOREM I: ADDITION. *Two power series may be added together for all values of x for which both series are convergent.* That is, if the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

are both convergent, the series obtained by adding these together will converge to the value $\phi(x) + \psi(x)$:

$$\phi(x) + \psi(x) = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots.$$

THEOREM II: MULTIPLICATION. *Two power series may be multiplied together for all values of x for which both series are absolutely convergent.*

That is, if the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

are both absolutely convergent, then

$$\phi(x) \cdot \psi(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots.$$

THEOREM III: DIVISION. *One convergent power series may be divided by another, provided the constant term in the denominator is not* 0. The interval of convergence of the quotient-series cannot in general be determined by elementary methods.

THEOREM IV: DIFFERENTIATION. *A power series may be differentiated term by term for all values of x within its interval of convergence*.*

THEOREM V: INTEGRATION. *A power series may be integrated term by term between any limits lying within the interval of convergence*.*

These theorems enable us to obtain many power developments in which the evaluation of the successive derivatives would be very tedious, and in which the law of formation of the coefficients is so complicated that the interval of convergence could hardly be determined directly.

Examples: (*a*) Expand $\sin^2 x$ in powers of x to x^6 inclusive. We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$= x - \tfrac{1}{6} x^3 + \tfrac{1}{120} x^5 - \cdots.$$

Squaring the trinomial and discarding all terms after x^6, we find

$$\sin^2 x = x^2 - \tfrac{1}{3} x^4 + \tfrac{2}{45} x^6 + \cdots.$$

By Theorem II, this series is seen to converge for all values of x.

(*b*) Find the area under the curve

$$y = \frac{\sin x}{x}$$

from $x = 0$ to $x = 1$.

The area is

$$A = \int_0^1 y\, dx = \int_0^1 \frac{\sin x}{x}\, dx.$$

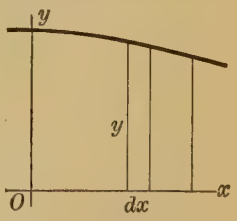


FIG. 88

The integral here occurring cannot be evaluated in terms of elementary functions. But we have

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots,$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots,$$

*The endpoints of the interval are excluded

whence (Theorem V)

$$A = \int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots\right) dx$$
$$= \left(x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \cdots\right)\Big]_0^1$$
$$= 1 - 0.05556 + 0.00167 - 0.00003 = 0.9461.$$

EXERCISES

Expand the following functions in Maclaurin's series by use of the theorems of § 153, and determine the interval of convergence in each case*.

1. $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$. *Ans.* $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$, all values.

2. $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. *Ans.* $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, all values.

3. $\log \frac{1+x}{1-x}$. *Ans.* $2(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots)$, $-1 < x < 1$.

4. Extend the series for $\sin^2 x$ to x^{10} inclusive.
Ans. $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \frac{2}{14175}x^{10} - \cdots$.

5. $\cos^2 x$. Cf. example (a) above.

6. $\sin^3 x$. *Ans.* $x^3 - \frac{1}{2}x^5 + \frac{13}{120}x^7 + \cdots$, all values.

7. $\cosh^2 x$. *Ans.* $1 + x^2 + \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$, all values.

8. $\sinh^2 x$. Cf. Ex. 7.

9. $e^x \sin x$. *Ans.* $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots$, all values.

10. $e^x \cos x$. *Ans.* $1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$, all values.

11. $e^x\sqrt{1-x^2}$. *Ans.* $1 + x - \frac{1}{3}x^3 - \frac{1}{3}x^4 + \cdots$, $-1 < x < 1$.

12. $\frac{e^x}{1-x}$. *Ans.* $1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \cdots$, $-1 < x < 1$.

13. $\frac{\log(1+x)}{1+x}$. *Ans.* $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \cdots$, $-1 < x < 1$.

14. $\sec x$. *Ans.* $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

15. $\tan x$. *Ans.* $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

16. $\tan^2 x$. *Ans.* $x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \cdots$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

17. By means of the formula $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, solve example ($a$) above and discover the law of formation of the terms.

* Where Theorem III is used, the interval cannot be found by the student.

18. By means of series, verify the formula $\sin 2x = 2 \sin x \cos x$.

19. Show that for values of x so small that the fourth and higher powers of $\frac{x}{a}$ may be neglected, the catenary

$$y = a \cosh \frac{x}{a} = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

may be replaced by a parabola.

20. Derive the series for $\cos x$ from the series for $\sin x$ by differentiation.

21. Derive the series for $\cos x$ from the series for $\sin x$ by integration.

22. Investigate the result of term-by-term differentiation and integration of the series for e^x.

23. Derive the series for $\cosh x$ from that for $\sinh x$ in two ways.

24. By differentiating the Maclaurin series for $\log (1 - x)$ (Ex. 7, p. 250), verify the formula of elementary algebra for the sum of an infinite geometric series.

25. Expand $\frac{1}{1 + x}$ in series by division (or by the binomial theorem), and integrate term by term. Cf. Ex. 6, p. 249.

26. By means of series, verify the formula

$$\frac{d}{dx}(\sec x) = \sec x \tan x.$$

27. Expand $\frac{1}{1 + x^2}$ by division, and integrate term by term.

Ans. $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots, \quad -1 \leqq x \leqq 1.$

28. Integrate the series for $\frac{1}{1 - x^2}$. Cf. Ex. 3.

29. Expand $\arcsin x$ in powers of x by integrating the binomial expansion of $\frac{1}{\sqrt{1 - x^2}}$.

Ans. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots, \quad -1 < x < 1.$

30. Derive the Maclaurin series for $\sec^2 x$ by four methods.

31. Integrate the series in Ex. 15.

Ans. $\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$

32. Find the interval in which $\tan \theta$ may be replaced by θ with an error not greater than 0.00005. *Ans.* $\theta < 3°$.

33. Within what interval may $\sin \theta$ be replaced by $\tan \theta$, with an error less than 0.0005? *Ans.* $\theta < 5° 43'$.

34. Compute (*a*) $\tan 2°$, (*b*) $\tan 6°$, to four decimal places.

35. Find the area under the curve $y = e^{-\frac{1}{2}x^2}$, from $x = 0$ to $x = 1$. *Ans.* 0.86.

36. Find the area under the curve $y = e^{-x^3}$, from $x = 0$ to $x = 1$. *Ans.* 0.81.

37. The area in Fig. 88 revolves about Ox. Find the volume generated. *Ans.* $0.90\,\pi$.

38. Find the area under the curve $y = x \sec x$ from $x = 0$ to $x = \frac{1}{2}$. *Ans.* 0.13.

39. By setting $x = \frac{1}{2}$ in the series for arcsin x (Ex. 29), compute the value of π to two decimal places.

40. Sum the series $1 - \frac{1}{3} + \frac{1}{5} - \cdots$. (Cf. Ex. 27.)

41. In leveling, error is introduced due to the curvature of the earth. Find the correction for 1 mile. *Ans.* 8 in.

42. In Ex. 41, find the correction for 2 miles. *Ans.* 2 ft. 8 in.

43. Two ships have masts reaching 80 ft. above the water level. How far is each masthead visible from the other? *Ans.* About 22 mi.

44. How much of the earth's surface is visible (theoretically) to an aviator at a height of one mile? *Ans.* About 25000 sq. mi.

45. Find the difference between the circumference of the earth and the perimeter of a regular circumscribed polygon of 1,000,000 sides. Cf. Ex. 43, p. 251.

46. Find the length of the arc of the parabola $y^2 = 4x$ from $x = 1000$ to $x = 2000$. (Expand $\sqrt{1 + \frac{1}{x}}$ in powers of $\frac{1}{x}$ by the binomial theorem.) *Ans.* 1000.347.

47. Show that the length of the arc of the hyperbola $xy = 1$ from $x = 100$ to $x = 200$ differs from the length of the chord by one part in 5,000,000,000.

154. The value of π. The series used in Ex. 39 above, for the determination of π converges too slowly to be available for the computation of a large number of decimal places. Many more convenient series have been derived; one of the simplest is the following.

By Ex. 27, p. 64, we have

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

whence (Ex. 27, p. 255)

$$(1)\quad \pi = 4\left(\frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \cdots\right) + 4\left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \cdots\right).$$

The value of π has actually been computed, by means of very rapidly convergent series, to 707 decimal places. Such a result, however, is of little practical value (see Ex. 8 below).

155. Computation of logarithms. The series (Exs. 6, 7, pp. 249–250)

$$(1)\qquad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

$$(2)\qquad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

converge rapidly for small values of x, so that they can be used to compute logarithms of numbers near unity. A more generally useful series may be derived as follows.

By Ex. 3, p. 254,

$$\log\frac{1+x}{1-x} = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots),\quad -1 < x < 1.$$

Let us put

$$\frac{1+x}{1-x} = \frac{m+n}{m},$$

or

$$x = \frac{n}{2m+n},$$

where m may have any positive value, and n any value such that $m + n > 0$. Then

$$(3)\quad \log(m+n) = \log m + 2\left(\frac{n}{2m+n}\right) + \frac{2}{3}\left(\frac{n}{2m+n}\right)^3 + \frac{2}{5}\left(\frac{n}{2m+n}\right)^5 + \cdots.$$

For values of m and n such that $\dfrac{n}{2m+n}$ is small, this series converges rapidly, and is well adapted to computation.

Example: (*a*) Taking $m = 1$, $n = 1$ in (3), we have

$$\log 2 = \frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3^3} + \frac{2}{5}\cdot\frac{1}{3^5} + \cdots.$$

$$\begin{aligned}
\frac{2}{3} &= 0.6666667\\
\frac{2}{3}\cdot\frac{1}{3^3} &= 0.0246914\\
\frac{2}{5}\cdot\frac{1}{3^5} &= 0.0016461\\
\frac{2}{7}\cdot\frac{1}{3^7} &= 0.0001306\\
\frac{2}{9}\cdot\frac{1}{3^9} &= 0.0000113\\
\frac{2}{11}\cdot\frac{1}{3^{11}} &= 0.0000010\\
\frac{2}{13}\cdot\frac{1}{3^{13}} &= 0.0000001\\
\log 2 &= 0.693147.
\end{aligned}$$

From this the logarithms of 4, 8, ⋯ may be found directly.

With $m = 2$, $n = 1$, we find $\log 3$; from this and the previous result we may obtain the logarithms of all numbers whose only prime factors are 2 and 3. In fact, only the logarithms of prime numbers need be computed by the series.

Since $\log m$ appears in the right member of (3), it is obvious that if $\log (m + n)$ is to be computed to k places, $\log m$ must be known to at least $k + 1$ places. For this reason, in using the series it is well to begin by computing the logarithms of several small integers — say 2, 3, 5, 7 — to considerably greater accuracy than is required for the later work. Then, to find the logarithm of any prime number $m + n$, we can always choose m and n subject to the following considerations: first, that m shall be a composite number whose only prime factors are the small integers referred to, so that $\log m$ is known; second, that $\frac{n}{2m + n}$ shall be small, so that the series converges rapidly.

For use in Exs. 29–41 below, we list the following logarithms:

$$\log 2 = 0.693147, \qquad \log 3 = 1.098612,$$
$$\log 5 = 1.609438, \qquad \log 7 = 1.945910.$$

Example: (*b*) Compute log 379.

An obvious choice of m and n is

$$m = 400 = 2^4 \cdot 5^2, \; n = -21.$$

Then

$$\log 379 = 4 \log 2 + 2 \log 5 - 2\left(\tfrac{21}{779}\right) - \tfrac{2}{3}\left(\tfrac{21}{779}\right)^3 - \cdots, \text{ etc.}$$

EXERCISES

1. Use the series (1), § 154, to compute π to four places.

2. Prove that

$$2 \arctan \frac{1}{3} + \arctan \frac{1}{7} = \frac{\pi}{4},$$

and derive a series for π.

3. Prove that

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4},$$

and derive a series for π.

4. Expand $(\arcsin z)^2$ in powers of z. (Cf. Ex. 29, p. 255.)

$$Ans.\; 2\left(\frac{z^2}{2} + \frac{2}{3} \cdot \frac{z^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\frac{z^6}{6} + \cdots\right), \; -1 < z < 1.$$

5. Expand $\dfrac{\arcsin z}{\sqrt{1 - z^2}}$ by differentiating the result of Ex. 4.

$$Ans.\; z + \frac{2}{3}z^3 + \frac{2 \cdot 4}{3 \cdot 5}z^5 + \cdots, \; -1 < z < 1.$$

6. By substituting

$$z = \frac{x}{\sqrt{1 + x^2}}$$

in Ex. 5, show that

$$\arctan x = \frac{x}{1 + x^2}\left[1 + \frac{2}{3} \cdot \frac{x^2}{1 + x^2} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{x^2}{1 + x^2}\right)^2 + \cdots\right].$$

7. Using the result of Ex. 6, show that the series (1), § 154, may be put in the form

$$\pi = \frac{16}{10}\left[1 + \frac{2}{3} \cdot \frac{2}{10} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{2}{10}\right)^2 + \cdots\right] + \frac{12}{10}\left[1 + \frac{2}{3} \cdot \frac{1}{10} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{1}{10}\right)^2 + \cdots\right].$$

8. The distance from the sun to Neptune is about 2,750,000,000 miles. Find the error in computing the volume of a sphere with center at the sun large enough to include the entire solar system, if the value used for π is correct to 50 decimal places, with an error of not more than 1 point in the 51st place. *Ans.* Less than $\frac{1}{100,000,000}$ cu. in.

9. Find log 4, log 8, log 64 to five places.

10. Find log 3 to five places.

11. Find log 5 to four places.

12. Find log 7 to four places, taking $m = 8$, $n = -1$.

Find the following logarithms to five places.

13. log 84.	**14.** log 378.	**15.** log 735.
16. log 1372.	**17.** log 2000.	**18.** log 2058.
19. log 2352.	**20.** log 6804.	**21.** log 8192.

22. Having found log 3, obtain log 10 by putting $x = \frac{1}{10}$ in (2), § 155.

23. Find $\log_{10} e$. **24.** Find $\log_{10} 2$.

25. Find $\log_{10} 3$. **26.** Find $\log_{10} 7$.

27. Find log 103 by means of (1), § 155. *Ans.* 4.6347.

28. Find log 997 by (2), § 155. *Ans.* 6.9048.

Being given the logarithms of 2, 3, 5, 7, compute the following logarithms by (3), § 155.

29. log 59.	*Ans.* 4.0775.	**30.** log 127.	*Ans.* 4.8442.
31. log 1.37.	*Ans.* 0.3148.	**32.** log 157.	*Ans.* 5.0562.
33. log 173.	*Ans.* 5.1533.	**34.** log 263.	*Ans.* 5.5722.
35. log 3.37.	*Ans.* 1.2149.	**36.** log 0.577.	*Ans.* -0.5499.
37. log 659.	*Ans.* 6.4907.	**38.** log 0.769.	*Ans.* -0.2627.
39. log 1327.	*Ans.* 7.1907.	**40.** log 7.621.	*Ans.* 2.0309.

41. Complete the computation in example (*b*), § 155.

42. In example (*b*), § 155, discover several choices of m and n more favorable than the one there adopted.

Assuming that the logarithms of 2, 3, 5, 7 are known, indicate suitable choices of m and n for computing the following logarithms.

43. log 547.	**44.** log 811.	**45.** log 883.
46. log 1103.	**47.** log 1301.	**48.** log 2179.
49. log 2903.	**50.** log 5281.	**51.** log 5669.
52. log 6637.	**53.** log 8287.	**54.** log 8501.
55. log 8747.	**56.** log 8831.	**57.** log 9421.

156. Taylor's series. We often wish to expand a function in powers of $x - a$, where a is some given number. Assuming

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots,$$

and setting $x = a$, we find $f(a) = c_0$. Proceeding as in § 150, we obtain finally *Taylor's series:*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots.$$

Thus Maclaurin's series is a special case of Taylor's series, *viz.* the case $a = 0$.

The remarks made in § 150 concerning the validity of Maclaurin's series apply here as well.

Maclaurin's series converges rapidly, in general, for *small* values of x. Taylor's series converges rapidly for values of x near a—*i.e.* such that $x - a$ is small. Hence, of the available values of a (*viz.* those for which $f(a)$, $f'(a)$, $\cdots$ are known), we should as a rule choose the one that is nearest the value of x for which $f(x)$ is to be computed. For example, to compute sin 46°, we would take $a = \frac{1}{4}\pi$; to compute sin 58°, we would take $a = \frac{1}{3}\pi$.

Example: Expand the function $\log x$ in powers of $x - 1$. In this case $a = 1$:

$$\begin{aligned} f(x) &= \log x, & f(1) &= 0, \\ f'(x) &= \frac{1}{x}, & f'(1) &= 1, \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -1, \\ f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2, \\ \vdots & \quad \vdots & \vdots & \quad \vdots \end{aligned}$$

Hence, by (1),

$$\log x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \cdots.$$

It is easily shown (Ex. 19, p. 245) that the series converges in the interval $0 < x \leqq 2$.

157. Approximate formulas for Δy *. In (1), § 156, let us replace a by x and x by $x + \Delta x$. The formula then becomes

$$(1)\quad f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)}{2!}\overline{\Delta x}^2 + \frac{f'''(x)}{3!}\overline{\Delta x}^3 + \cdots,$$

or, with

$$y = f(x), \quad \Delta y = f(x+\Delta x) - f(x),$$

$$(2)\qquad \Delta y = y'\,\Delta x + \frac{y''}{2!}\overline{\Delta x}^2 + \frac{y'''}{3!}\overline{\Delta x}^3 + \cdots.$$

For values of x and Δx that cause the terms of the series to diminish rapidly, we have, as a *first approximation* to the value of Δy, the formula used in § 60, viz.

$$\Delta y = y'\,\Delta x = dy,$$

the error in using this formula being approximately $\frac{1}{2}\,y''\overline{\Delta x}^2$; as a *second approximation* we have

$$\Delta y = y'\,\Delta x + \tfrac{1}{2}\,y''\overline{\Delta x}^2,$$

with an error nearly equal to $\frac{1}{6}\,y'''\overline{\Delta x}^3$; etc.

Example: Taking $f(x) = \cos x$ in formula (1), we get

$$\cos(x+\Delta x) = \cos x - \sin x\,\Delta x - \tfrac{1}{2}\cos x\,\overline{\Delta x}^2 + \cdots,$$

which gives the first approximation

$$(3)\quad \cos(x+\Delta x) = \cos x - \sin x\,\Delta x.$$

If the allowable error in the cosine is 0.0005, within what range may this formula be used to compute the cosines of angles near 45°†?

The error in using (3) is approximately $-\frac{1}{2}\cos x\,\overline{\Delta x}^2$. Since $x = 45°$, we have, numerically,

$$\tfrac{1}{2}\cdot 0.7071\,\overline{\Delta x}^2 < 0.0005,$$

$$\overline{\Delta x}^2 < 0.001414,$$

$$\Delta x < 0.0376 = 2°, \text{ roughly.}$$

Hence the approximate formula (3) may safely be used to compute the cosines of angles between 43° and 47°.

* Cf. § 60.

† Cf. Ex. 7(*b*), p. 94.

EXERCISES

1. Expand e^x in powers of $x - 2$.

Ans. $e^x = e^2\left[1 + (x-2) + \frac{1}{2!}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \cdots\right]$, all values.

2. Expand $\sin x$ in powers of $x - \frac{\pi}{4}$.

Ans. $\frac{1}{2}\sqrt{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots\right]$, all values.

3. Expand $\log x$ in powers of $x - 3$.

4. Expand $\sqrt{x}$ in powers of $x - 1$.

Ans. $1 + \frac{1}{2}(x-1) - \frac{1}{2^2}(x-1)^2 + \frac{3}{2^3}(x-1)^3 - \frac{3 \cdot 5}{2^4}(x-1)^4 + \cdots$, $0 < x < 2$.

5. Show that, if $P(x)$ is a polynomial of the n-th degree in x,

$$P(x) = P(a) + P'(a)(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \cdots + \frac{P^{(n)}(a)}{n!}(x-a)^n,$$

whatever may be the values of a and x.

6. Expand $\cos x$ in powers of $x - \frac{1}{3}\pi$. Use the result to compute $\cos 58°$, $\cos 61°$.

7. Using the series of Ex. 1, find $e^{2.05}$.

8. Expand e^x in powers of $x - 1$, and compute $e^{0.97}$. *Ans.* 2.6380.

9. Show that the error in using the approximate formula of example (a), § 60, is $\pi\overline{\Delta r}^2$. When $r = 10$ ft., what is the greatest allowable value of Δr if accuracy to 5% is required? *Ans.* About 1 ft.

10. Find the error in the approximate formula for the volume of a thin spherical shell (Ex. 2, p. 93). What is the greatest allowable thickness for a radius of 5 ft., if accuracy to 1% is required?

Ans. About 0.6 in.

11. Solve the example of § 157 for angles near 60°.

Ans. $57° 30' < x < 62° 30'$.

12. If the allowable error in the tangent is 0.00005, within what range can the approximate formula of Ex. 7(c), p. 94, be used?

Ans. $44° 43' < \theta < 45° 17'$.

13. Apply (1), § 157, to the function $\frac{1}{x}$. (Cf. Ex. 35, p. 88, and Ex. 15, p. 94.)

Ans. $\frac{1}{x + \Delta x} = \frac{1}{x} - \frac{\Delta x}{x^2} + \frac{\overline{\Delta x}^2}{x^3} - \frac{\overline{\Delta x}^3}{x^4} + \cdots$.

14. In a table of reciprocals, the allowable error is 0.00000005. In what range may we take (a) $\frac{1}{x+1} = \frac{1}{x}$? ($b$) $\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2}$? ($c$) $\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}$? (Cf. Ex. 13.)

Ans. (a) $x > 4472$; (b) $x > 271$; (c) $x > 67$.

15. Show that

$$(x + \Delta x)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{\Delta x}{2\,x^{\frac{1}{2}}} - \frac{\overline{\Delta x}^2}{8\,x^{\frac{3}{2}}} + \frac{\overline{\Delta x}^3}{16\,x^{\frac{5}{2}}} - \cdots.$$

16. In a table of square roots, the allowable error in the roots is 0.01. For what values of x may we use the approximate formula $\sqrt{x+1} = \sqrt{x} + \frac{1}{2\sqrt{x}}$? (Cf. Ex. 15; see also Ex. 20, p. 95.) *Ans.* $x > 5$.

17. Apply (1), § 157, to the function $\log x$.

Ans. $\log (x + \Delta x) = \log x + \frac{\Delta x}{x} - \frac{\overline{\Delta x}^2}{2\,x^2} + \frac{\overline{\Delta x}^3}{3\,x^3} - \cdots.$

18. The allowable error in a table of logarithms is 0.0000005. Within what range can the approximate formula of Ex. 28, p. 95, be used? *Ans.* $x > 1000$.

CHAPTER XXIII

PARTIAL DIFFERENTIATION

158. Functions of several variables. Up to this point we have been concerned with functions of a single argument. A function may, however, depend upon several independent variables. For example, the volume of a circular cylinder is a function of its radius and altitude; the acceleration of a moving particle is a function of all the forces acting on it; the strength of a rectangular beam is a function of its breadth and depth.

If z is a function of two variables x and y, we write

$$z = f(x, y),$$

with a similar notation for functions of more than two variables.

Geometrically a function of two variables x and y may be represented as the z-coördinate of a surface in space. Thus the equation

$$z = ax + by + c$$

represents a plane; the equation

$$z = x^2 - y^2$$

represents a hyperbolic paraboloid, etc.

A thorough study of functions of several variables is beyond the scope of a first course in the calculus. In the present chapter we set forth a few of the most important definitions and theorems, confining our attention chiefly to functions of two arguments.

159. Limits; continuity. Suppose we have given a function of two variables

$$z = f(x, y) \tag{1}$$

representing a surface in space. When x and y approach the respective values x_1, y_1, the function z is said to *approach a limit* z_1 if the point (x, y, z) of the surface (1) approaches a definite limiting point (x_1, y_1, z_1). In other words, if when x is sufficiently near x_1 and y is sufficiently near y_1 the difference between z and z_1 becomes and remains numerically less than any preassigned positive quantity however small, then z is said to approach the limit z_1: in symbols,

$$\lim_{\substack{x \to x_1 \\ y \to y_1}} f(x, y) = z_1.$$

A function $f(x, y)$ is said to be *continuous* at the point (x_1, y_1) if

$$\lim_{\substack{x \to x_1 \\ y \to y_1}} f(x, y) = f(x_1, y_1).$$

Similar definitions are laid down for functions of more than two variables.

In what follows, it is supposed that all functions occurring are continuous at all points under consideration.

160. Partial derivatives. If y be kept *fixed*, the function

$$z = f(x, y)$$

becomes a function of x alone, and its derivative may be found by the ordinary rules. This derivative is called the *partial derivative of z with respect to x*, and is denoted by any one of the symbols

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad f_x(x, y).$$

The partial derivative with respect to y has a similar meaning.

The idea of partial differentiation may be extended at once to functions of any number of variables. We have only to remember that in differentiating with respect to any one variable, *all the other variables are treated as constants*

161. Geometric interpretation of partial derivatives. To keep y constant, say $y = y_1$, in the equation $z = f(x, y)$ means geometrically that we cut the surface by the plane $y = y_1$. The partial derivative $\frac{\partial z}{\partial x}$ is therefore the slope of the curve of intersection of the surface and the plane $y = y_1$.

The partial derivative $\frac{\partial z}{\partial y}$ may be interpreted similarly.

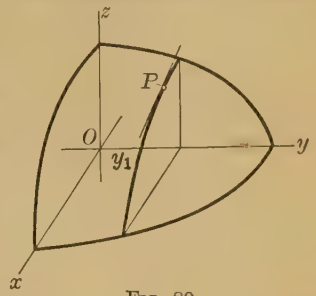


FIG. 89

162. Higher derivatives. The derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are themselves functions of x and y, and their partial derivatives can in turn be found. They are denoted by the following symbols:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y).$$

The process can of course be repeated to find still higher derivatives.

It can be shown that the two "cross-derivatives" $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$ are identical:

$$\frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial^2 z}{\partial x \partial y}.$$

That is, *the order of differentiation is immaterial.* This is true for derivatives of all orders, and for functions of any number of variables.

EXERCISES

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ for the following functions.

1. $z = x^2 + xy - 3x + 5$. *Ans.* $\frac{\partial z}{\partial x} = 2x + y - 3$; $\frac{\partial z}{\partial y} = x$.
2. $z = x^3 + 3x^2y - xy + 2y - 3$.
3. $z = (x^2 - 2y^2)^3$.
4. $z = \frac{x - y}{x + y}$.
5. $z = \log\sqrt{x^2 + y^2}$.
6. $z = \arctan\frac{y}{x}$. *Ans.* $\frac{\partial z}{\partial x} = \frac{-y}{x^2 + y^2}$.
7. $z = e^{xy}$.
8. $z = \sin^2(x - y)$.
9. $z = e^{\cos^2(x-y)}$.
10. Find the slope of the curve cut from the hyperbolic paraboloid $z = x^2 - 2y^2$ by the plane $y = 3$, at the point $(4, 3, -2)$. *Ans.* 8.
11. Find the equations of the tangent to the parabola
$$z = 3x^2 + 4y^2,\ x = 2$$
at the point $(2, 1, 16)$. *Ans.* $x = 2,\ z = 8y + 8$.
12. Find the tangent to the circle $x^2 + y^2 = a^2,\ z = a$ at $(\frac{3}{5}a, \frac{4}{5}a, a)$.
Ans. $3x + 4y = 5a,\ z = a$.
13. Given $f(x, y, z) = xyz + 3x^2y + z^3$, find f_x, f_y, f_z.
Ans. $f_x = y(z + 6x)$.
14. If $u = x^2 + y^2 - z^2 + x + 2y$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.
15. If $u = x^3y^2 - 2xy^4 + 3x^2y^3$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u$.
16. If $u = (y - z)(z - x)(x - y)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
17. Given $z = x + x^3y^2 + 2x^2y^4$, find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$.
Ans. $\frac{\partial^2 z}{\partial x^2} = 6xy^2 + 4y^4$; $\frac{\partial^2 z}{\partial y \partial x} = 6x^2y + 16xy^3$.
18. Given $z = x^2y^2 + 3x^3y^3 - x^2y$, verify that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.
19. If $z = \cos(x - y)$, show that $\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial x^2 \partial y}$.
20. If $z = \frac{1}{2}\log(x^2 + y^2)$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.
21. If $z = x^2y$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$.
22. If $u = x^2 + y^2 + yz$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$.

163. Total differentials. When x and y change by amounts Δx and Δy, the function $z = f(x, y)$ changes by an amount Δz. It can be shown that Δz may be expressed in the form

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon\Delta x + \eta\Delta y,$$

where ϵ and η are infinitesimals.

The quantity $\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ is called the *principal part* (cf. § 58) of the infinitesimal Δz, or the *total differential of* z:

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

In particular, if $z = x$, $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 0$, so that

$$dx = \Delta x.$$

Similarly

$$dy = \Delta y.$$

Hence we may write

(1) $$\boldsymbol{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.}$$

If x and y are functions of a third variable t, then z becomes a function of t alone, and its differential has been defined in § 59. It can be shown that the value of dz as given by (1) agrees with the earlier definition, and in fact, that (1) is true *no matter what the independent variables may be.*

For functions of more than two arguments a similar formula holds. Thus, if

$$u = f(x, y, z),$$

the total differential of u is

(2) $$\boldsymbol{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.}$$

This formula likewise is valid for any system of independent variables.

Example: If $z = x^2 - xy + y^2$,
then $dz = 2x\,dx - x\,dy - y\,dx + 2y\,dy.$

164. Approximate formulas. Approximate formulas analogous to those of § 60 can be found for functions of two variables by means of (1), § 163.

Example: Find the error in the value of a fraction produced by small errors in the numerator and denominator.

Let the fraction be

$$f = \frac{y}{x}. \tag{1}$$

Then the change in f due to small changes in y and x is approximately (unless x is small)

$$df = \frac{x\,\Delta y - y\,\Delta x}{x^2}.$$

165. Differentiation of implicit functions. Let y be defined as a function of x by the equation

$$f(x, y) = 0.$$

Let us for an instant put

$$z = f(x, y);$$

then by (1), § 163,

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

But in the present instance

$$z = 0,$$

hence

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0,$$

or

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \left(\frac{\partial f}{\partial y} \neq 0\right). \tag{1}$$

The value of $\frac{dy}{dx}$ as given by this formula is of course identical with that given by the method of § 21.

Again, let z be defined implicitly as a function of the two independent variables x and y by the equation

$$F(x, y, z) = 0.$$

Put

$$u = F(x, y, z);$$

then

$$du = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz.$$

But since $u = 0$, $du = 0$ likewise, and

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0.$$

Further, since z is a function of x and y, we may write

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Eliminating dz between these two equations, we find

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right)dy = 0.$$

To find $\dfrac{\partial z}{\partial x}$, keep y fixed, so that $dy = 0$. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,$$

or

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \left(\frac{\partial F}{\partial z} \neq 0\right). \tag{2}$$

Similarly

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} \qquad \left(\frac{\partial F}{\partial z} \neq 0\right). \tag{3}$$

EXERCISES

Find the total differential of each of the following functions.

1. $z = x^3 - 2x^2y + 1$. *Ans.* $dz = 3x^2\,dx - 2x^2\,dy - 4xy\,dx$.

2. $z = x^2 - 3xy + y^2 + 2x - y + 1$.

3. $z = \log\sqrt{x^2 + y^2}$. *Ans.* $dz = \dfrac{x\,dx + y\,dy}{x^2 + y^2}$.

4. $z = \dfrac{x^2}{y}$. *Ans.* $dz = \dfrac{2xy\,dx - x^2\,dy}{y^2}$.

5. $r = \cos^2(x - y)$.

Ans. $dr = 2\cos(x - y)\sin(x - y)(dy - dx)$.

6. $u = x^2 + y^2 - z^2$.

7. $u = x^3y^2z$.

8. $u = \dfrac{x^2 - y^2}{z^2}$. *Ans.* $du = \dfrac{2z(x\,dx - y\,dy) - 2(x^2 - y^2)dz}{z^3}$.

9. $z = \sqrt{x^2 + y^2}$. *Ans.* $dz = \dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$.

10. $\theta = \arctan\dfrac{y}{x}$. *Ans.* $d\theta = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

11. $z = \dfrac{y}{\sqrt{x^2 + y^2}}$. *Ans.* $dz = \dfrac{x(x\,dy - y\,dx)}{(x^2 + y^2)^{\frac{3}{2}}}$.

12. $u = \sqrt{x^2 + y^2 + z^2}$.

13. $z = \log\sqrt{\dfrac{y}{x}}$.

14. $u = \log\dfrac{x^3y^2}{2z}$.

15. $u = e^{x-y}$.

16. $u = \sinh(x + y)$.

17. Find the error in the product of two numbers due to small errors in the numbers. *Ans.* $x\Delta y + y\Delta x$.

18. A lot is approximately 50×100 ft. If each of the measurements may be inaccurate by as much as 3 in., find the possible error in the area. *Ans.* 37.5 sq. ft.

19. The radius and altitude of a circular cylinder are found by measurement to be $r = 6.00 \pm 0.04$ ft., $h = 9.00 \pm 0.03$ ft. Find the volume. *Ans.* $\pi(324 \pm 5.40)$ cu. ft.

20. In Ex. 19, find the total surface. *Ans.* $\pi(180 \pm 2.04)$ sq. ft.

21. The sides of a rectangle are 4.00 ± 0.01, 8.00 ± 0.02. Find the moment of inertia with respect to the longer side.

Ans. 170.67 ± 1.71.

22. In Ex. 19, find the moment of inertia with respect to the axis.

Ans. $\pi(5832 \pm 175.0)$ ft^5.

23. In Ex. 19, find the moment of inertia with respect to the base.

Ans. $\pi(8748 \pm 204.1)$ ft^5.

24. A right angle is constructed and the tangent of one of the acute angles is found by measuring the opposite side y and the adjacent side x. If $y = 2.5 \pm 0.1$, $x = 1.5 \pm 0.1$, what is the greatest possible error in the tangent? (Cf. the example of § 164.) *Ans.* About 0.18.

25. It is desired to draw a line through the points (0, 0), (3, 4). If the line is actually drawn through $(-0.04, 0.11)$ and (3.01, 3.94), find the error in the slope of the line. *Ans.* -0.08.

26. A sprinter runs 100 yd. in about 10 sec. If the time may be in error by as much as $\frac{1}{10}$ sec., and the distance by as much as 6 in., find the possible error in the velocity. *Ans.* 0.35 ft. per sec.

27. Find the change in the hypotenuse of a right triangle caused by small changes in the sides. (Cf. Ex. 9.)

28. In Ex. 24, find the greatest possible error in the angle. (Cf. Ex. 10.) *Ans.* About 2° 42′.

29. In Ex. 24, find the greatest possible error in the sine of the angle. (Cf. Ex. 11.) *Ans.* 0.024.

30. Find the error in the common logarithm of the product of two numbers caused by small errors in the numbers.

Ans. $0.4343\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right)$.

31. The dimensions x, y, z of a rectangular parallelepiped change by small amounts Δx, Δy, Δz. Find dV, ΔV.

32. A box is approximately $2 \times 4 \times 5$ ft. If the dimensions are accurate to 0.1 in., find the possible error in the volume.

Ans. About 547 cu. in.

33. The dimensions of a box are 1 ± 0.01, 4 ± 0.02, 8 ± 0.03. Find the length of a diagonal. *Ans.* 9 ± 0.04.

34. In Ex. 33, find the amount of sheet metal necessary to make the box. *Ans.* 88 ± 0.90 sq. units.

35. Find the uncertainty in the logarithm of the product of three numbers due to slight uncertainty in the numbers.

Find $\frac{dy}{dx}$ in the following cases, using formula (1) of § 165.

36. $3x - 4y + 2xy = 1$.

37. $(2x^2 - 3y^2)^2 + 1 - xy = 0$.

38. $x^3 + y^3 - 3axy = 0$.

39. $x + \log(x^2 + y^2) = 1$.

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in the following cases, using formulas (2), (3) of § 165.

40. $x^2 + y^2 + z^2 = 1$. *Ans.* $\frac{\partial z}{\partial x} = -\frac{x}{z}$, $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

41. $2x^2 - y^2 + 3z^2 = 1$.

42. $x + y - 2z^2 = z$. *Ans.* $\frac{\partial z}{\partial x} = \frac{1}{1 + 4z}$.

43. $x^2 - xy + yz = 0$.

166. Tangent plane and normal line. It can be shown that all the lines tangent to a surface

$$z = f(x, y)$$

at a point P: (x_1, y_1, z_1) lie in a plane *, the *tangent plane* to the surface at that point. This plane is of course determined by any two of the tangent lines. Let us assume the equation of the tangent plane in the form

$$z - z_1 = m_1(x - x_1) + m_2(y - y_1),$$

where m_1 and m_2 are to be determined. Now the line of intersection of this plane with the plane $y = y_1$ has the slope m_1. But this line is the tangent lying in the plane $y = y_1$, and, by § 161, its slope is the value of $\frac{\partial z}{\partial x}$ at P, which value we shall denote by the symbol $\left.\frac{\partial z}{\partial x}\right]_P$: that is,

$$m_1 = \left.\frac{\partial z}{\partial x}\right]_P.$$

Similarly we find

$$m_2 = \left.\frac{\partial z}{\partial y}\right]_P.$$

Thus the equation of the plane tangent to the surface

$$z = f(x, y)$$

at (x_1, y_1, z_1) is

$$z - z_1 = \left.\frac{\partial z}{\partial x}\right]_P (x - x_1) + \left.\frac{\partial z}{\partial y}\right]_P (y - y_1). \tag{1}$$

More generally, let the equation of the surface be given in the implicit form

$$F(x, y, z) = 0, \tag{2}$$

where the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ do not all vanish at P: (x_1, y_1, z_1). Suppose for definiteness that $\left.\frac{\partial F}{\partial z}\right]_P \neq 0$.

* Provided z, $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$ are continuous at P.

We may imagine equation (2) solved for z, and may then write the equation of the tangent plane by formula (1). But, by (2) and (3) of § 165,

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}.$$

Substituting these values in (1), we find

$$z - z_1 = -\frac{\dfrac{\partial F}{\partial x}\Big]_P}{\dfrac{\partial F}{\partial z}\Big]_P}(x - x_1) - \frac{\dfrac{\partial F}{\partial y}\Big]_P}{\dfrac{\partial F}{\partial z}\Big]_P}(y - y_1),$$

or

$$(3) \quad \frac{\partial F}{\partial x}\Big]_P (x - x_1) + \frac{\partial F}{\partial y}\Big]_P (y - y_1) + \frac{\partial F}{\partial z}\Big]_P (z - z_1) = 0.$$

The *normal* to a surface at a point P is the line through P perpendicular to the tangent plane.

It will be recalled from solid analytic geometry that the direction cosines of any line perpendicular to the plane

$$Ax + By + Cz + D = 0$$

are *proportional to the coefficients* A, B, C. Hence, since the normal is perpendicular to the tangent plane (3), we have at once the following

THEOREM: *The direction cosines of the normal to the surface*

$$F(x, y, z) = 0$$

at any point are proportional to the values of $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$, $\dfrac{\partial F}{\partial z}$ *at that point.*

This theorem is fundamental in the geometry of surfaces.

By analytic geometry, the equations of a line through x_1, y_1, z_1 with direction cosines proportional to a, b, c are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

From this the equations of the normal at any point may be written down at once.

Example: Find the tangent plane and normal line to the hyperboloid $2x^2 - y^2 + 3z^2 = 1$ at $(1, -2, -1)$.

Here

$$\frac{\partial F}{\partial x} = 4x, \qquad \left.\frac{\partial F}{\partial x}\right]_P = 4,$$

$$\frac{\partial F}{\partial y} = -2y, \qquad \left.\frac{\partial F}{\partial y}\right]_P = 4,$$

$$\frac{\partial F}{\partial z} = 6z, \qquad \left.\frac{\partial F}{\partial z}\right]_P = -6.$$

Hence the tangent plane is

$$4(x - 1) + 4(y + 2) - 6(z + 1) = 0,$$

or

$$2x + 2y - 3z = 1;$$

the normal is

$$\frac{x - 1}{2} = \frac{y + 2}{2} = \frac{z + 1}{-3}.$$

167. Angle between two surfaces; between a line and a surface. The angle between two surfaces at a point of intersection is defined as the *angle between the tangent planes* at that point, and this in turn is equal to the angle between the normals. This angle may be found by the theorem of analytic geometry that, if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively, the angle between them is given by the formula

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2.$$

The angle at which a line pierces a surface is defined as the *angle between the line and the tangent plane* at the piercing-point. This is evidently the complement of the angle between the line and the normal.

Example: Find the angle between the cylinder $y^2 = 4x$ and the ellipsoid $2x^2 + y^2 + z^2 = 7$ at the point $(1, 2, 1)$.

For the ellipsoid, the partial derivatives are

$$\frac{\partial F}{\partial x} = 4x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 2z,$$

hence the direction cosines of the normal at (1, 2, 1) are proportional to 4, 4, 2, and their actual values are $\frac{2}{3}$, $\frac{2}{3}$, $\frac{1}{3}$. For the cylinder,

$$\frac{\partial F}{\partial x} = -4, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 0,$$

hence the direction cosines of the normal at the point (1, 2, 1) are $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 0. Therefore

$$\cos\theta = -\frac{1}{\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = 0:$$

the surfaces intersect at right angles.

EXERCISES

Find the equations of the tangent plane and normal line to each of the following surfaces at the point indicated.

1. The cone $x^2 + 3y^2 = z^2$ at (2, 2, 4); draw the figure.

   Ans. $x + 3y - 2z = 0$; $\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-4}{-2}$.

2. The paraboloid $z = x^2 - y^2$ at (1, 1, 0).

   Ans. Tangent plane: $2x - 2y - z = 0$.

3. The ellipsoid $3x^2 + y^2 + z^2 = 16$ at (1, 2, -3).

   Ans. Tangent plane: $3x + 2y - 3z = 16$.

4. The hyperboloid $x^2 - y^2 - 2z^2 + 2 = 0$ at (3, -3, -1).

   Ans. Tangent plane: $3x + 3y + 2z + 2 = 0$.

5. The paraboloid $z = xy$ at (2, 3, 6).

   Ans. Normal: $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-6}{-1}$.

6. The paraboloid $x^2 + 2z^2 = 3y$ at (2, 4, -2).

   Ans. Tangent plane: $4x - 3y - 8z = 12$.

7. The surface $x^2y + y^2 + z^2 = 7$ at (1, 2, -1).

   Ans. Tangent plane: $4x + 5y - 2z = 16$.

8. The cylinder $y^2 = 4ax$ at $(a, 2a, a)$. Draw the figure.

   Ans. $x - y + a = 0$; $\frac{x-a}{1} = \frac{y-2a}{-1} = \frac{z-a}{0}$.

9. The paraboloid $yz = x$ at the origin.

   Ans. $x = 0$; $y = 0$, $z = 0$

10. The sphere $x^2 + y^2 + z^2 = a^2$ at (x_1, y_1, z_1).

    Ans. Tangent plane: $x_1x + y_1y + z_1z = a^2$.

11. Show that at any point on the z-axis there are two tangent planes to the surface $a^2y^2 = x^2(b^2 - z^2)$.

12. Show that the sum of the squares of the intercepts on the axes made by a tangent plane to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ is constant. Sketch this surface.

13. Prove that the tetrahedron formed by the coördinate planes and a tangent plane to the surface $xyz = a^3$ is of constant volume.

14. Find the angle between the sphere $x^2 + y^2 + z^2 = 14$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 20$ at the point $(-1, -2, 3)$.

Ans. $23° 33'$.

15. Show that the sphere $x^2 + y^2 + z^2 = 2a^2$ and the hyperbolic cylinder $xy = a^2$ are tangent to each other at the point $(a, a, 0)$.

16. Find the angle at which the normal to the hyperboloid $y^2 - x^2 + 4z^2 = 16$ at the point $(2, 2, 2)$ intersects the xy-plane. Draw the figure. *Ans.* $\arcsin \frac{2}{3}\sqrt{2}$.

17. Find the equations of the projections on the coördinate planes of the normal to the cylinder $x = y + z^2$ at $(2, 1, 1)$.

Ans. $2y = z + 1$, $2x + z = 5$, $x + y = 3$.

18. Determine a and b so that the ellipsoid $x^2 + 2y^2 + z^2 = 7$ and the paraboloid $z = ax^2 + by^2$ may intersect at right angles at $(1, 1, 2)$.

Ans. $a = 3$, $b = -1$.

19. Find the angle between the normal to the oblate spheroid $x^2 + y^2 + 2z^2 = 10$ at $(2, 2, 1)$ and the line joining the origin to that point. *Ans.* $\arccos \frac{5}{9}\sqrt{3}$.

20. In Ex. 19, find the shortest distance from the origin to the normal in question. *Ans.* $\frac{1}{3}\sqrt{6}$.

21. Find the angle at which the line $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ pierces the ellipsoid $2x^2 + 4y^2 + z^2 = 25$. *Ans.* $67° 49'$.

22. Prove that every line through the center of a sphere intersects the sphere at right angles.

CHAPTER XXIV

MULTIPLE INTEGRALS

168. Volume under a surface. The methods employed in Chapter XVIII for finding the volume of a solid succeed only when the solid can be cut into slices such that the area of the face of each slice is known. We proceed to develop a method that is free of this restriction.

Consider the volume V bounded by a portion T of the surface

$$z = f(x, y),$$

the area S into which T projects in the xy-plane, and the cylindrical surface through the boundaries of S and T.

We can get an approximate expression for this volume as follows. Draw in S a set of n lines parallel to the y-axis and a set of m lines parallel to the x-axis, as in Fig. 90, thus dividing S into rectangles of area $\Delta y \Delta x$, together with a number of irregular portions around the boundary. By passing through each line of the two sets a plane perpendicular to the xy-plane we divide V into vertical rectangular columns, together with smaller irregular columns. The upper boundary of each rectangular column is a portion of the surface T. Through that point of the upper boundary of each column which is

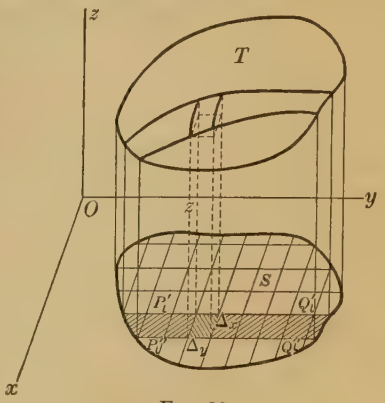


FIG. 90

nearest the xy-plane, pass a horizontal plane, thus forming a set of rectangular prisms lying wholly within V. The sum of the volumes of these prisms is evidently an approximation to the required volume, the error committed being the sum of the irregular columns around the outside, together with the portions lying above the upper bases of the rectangular prisms. That is, *approximately*,

$$V = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta y \Delta x,$$

where $f(x_i, y_j)$ is the altitude of the prism, and the double summation means that we cover all the full rectangles lying in S.

It is obvious that the error in this approximation may be made arbitrarily small by taking both Δx and Δy sufficiently small. Hence the required volume is *exactly*

$$V = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\ \Delta y\ \Delta x. \tag{1}$$

The " double limit " (1) may be evaluated by two successive applications of the fundamental theorem of § 114, as follows.

Let us fix our attention on the rectangle $P_i'P_i''Q_i''Q_i'$ in S (Fig. 90). The volume $\Delta V_i'$ whose base is this rectangle may be found approximately by adding the volumes of all the included elementary prisms. Hence, by § 114, $\Delta V_i'$ is given exactly by the formula

$$\begin{aligned} \Delta V_i' &= \lim_{m \to \infty} \sum_{j=1}^{m} f(x_i, y_j)\Delta y \Delta x \\ &= \left[\int_{y_i'}^{y_i''} f(x_i, y)\, dy \right] \Delta x, \end{aligned}$$

x_i and Δx remaining constant as we pass to the limit.

It is to be noticed that in the expression for $\Delta V_i'$ the coefficient of Δx is a function of x_i alone, since the limits y_i' and y_i'' are functions of x_i alone. Thus we may apply again the theorem of § 114, and find that the required volume under the

surface $z = f(x, y)$ is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\int_{y_i'}^{y_i''} f(x_i, y)\, dy \right] \Delta x$$

$$= \int_a^b \left[\int_{y'}^{y''} f(x, y)\, dy \right] dx,$$

where a and b are the extreme values of x on the boundary of S.

The quantity just found is usually written without the brackets, thus:

$$V = \int_a^b \int_{y'}^{y''} f(x, y) dy\, dx. \tag{2}$$

It is called a *double integral*, or more properly an *iterated integral*, being merely an integral of an integral. It is to be noted that *the inner integral sign belongs with the inner differential*, and that *during the integration with respect to* y, x *remains constant*. Further, the first or inner limits of integration are in general variables, but the outer limits are always constants.

Of course we might integrate first with respect to x, then with respect to y. The same argument as before would lead to the formula

$$V = \int_c^d \int_{x'}^{x''} f(x, y) dx\, dy, \tag{3}$$

y remaining constant during the first integration. The change from (2) to (3) is called *inverting the order* of integration.

In the foregoing argument, we have assumed our volume to be divided into rectangular columns perpendicular to the xy-plane. Frequently, however, it is more convenient to erect columns perpendicular to one of the other coördinate planes. Such variations offer no difficulty provided the geometric meaning of the successive integrations be kept clearly in mind. In every problem, a sketch of the required

volume should be made, and the required double integral built up by inspection of the figure.

It is hardly necessary to say that when the result is more easily found by simple integration, that method should be employed.

Any function $f(x, y)$ of two independent variables may be interpreted as the z-coördinate of a point on a surface in space. If, then, in any problem, we can express the required quantity as a double limit of the form (1), *no matter what may be the geometric or physical meaning of the given function* $f(x, y)$, the limit may be evaluated by an iterated integration (2) or (3). Thus the method described above is by no means confined to the determination of volumes — we shall apply it to the study of a variety of problems.

Example: Find the volume in the first octant bounded by the plane $z = x + y$ and the cylinder $y = 1 - x^2$.

Integrating in the order y, x, we have

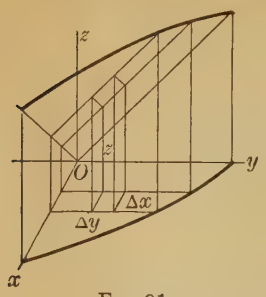


FIG. 91

$$V = \int_0^1 \int_0^{1-x^2} z \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} (x + y) dy \, dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \right)\Big]_0^{1-x^2} dx$$

$$= \int_0^1 [x - x^3 + \tfrac{1}{2}(1 - x^2)^2] \, dx$$

$$= \tfrac{3}{6}\tfrac{1}{0}.$$

EXERCISES

In each of the following exercises, the limits of integration should be obtained directly from a figure.

1. Find the volume in the first octant bounded by the planes $x = 1$, $z = x + y$ and the cylinder $y^2 = x$, integrating in the order y, x; check by integrating in the order x, y. *Ans.* $\frac{13}{20}$.

2. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x + y$. *Ans.* $\frac{2}{3} a^3$.

3. Find the volume in the first octant bounded by the surfaces $z = 1 - xy$, $y = 1 - x^2$. Solve in two ways. *Ans.* $\frac{7}{12}$.

4. Find in two ways the volume in the first octant bounded by the surfaces $az = x^2 + ay$, $y = x$, $x = a$. *Ans.* $\frac{5}{12}a^3$.

5. Find the volume of a cylindrical column standing on the area common to the two parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 1 + y - x^2$. Solve in two ways. *Ans.* $\frac{167}{420}$.

6. By two methods, find the volume in the first octant under the surface $az = xy$ and bounded by the surfaces $y^2 = ax$, $x = a$.

Ans. $\frac{1}{8}a^3$.

7. Find the volume bounded by the surfaces $y^2 + x + z = 4$, $y + z = 1$ and the coördinate planes. Solve in two ways. *Ans.* $\frac{7}{4}$.

8. Find the volume in the first octant bounded by the surfaces $az = xy$, $x^2 = 4ay$, $x - y = a$. *Ans.* $\frac{1}{24}a^3$.

9. Find the volume inclosed by the surfaces $az = y^2 + ax$, $y = x$, $x + y = 2a$, $x = 0$, $z = 0$. *Ans.* $\frac{3}{2}a^3$.

10. Find the volume in the first octant bounded by the surfaces $x^2 + az = a^2$, $x^2 + z^2 = ay$. Solve in two ways. *Ans.* $\frac{2}{7}a^3$.

11. Find the volume under the surface $az = xy$, whose base is the area in the xy-plane bounded by the curves $y^2 = ax$, $x + y = 2a$, $y = 0$.

Ans. $\frac{3}{8}a^3$.

12. Find the volume in the first octant bounded by the surfaces $x^2 = ay$, $x^2 = az$, $2x - y = a$. *Ans.* $\frac{11}{480}a^3$.

13. Find the volume sliced off from the paraboloid $az = a^2 - x^2 - y^2$ by the plane $y + z = a$. *Ans.* $\frac{3}{32}\pi a^3$.

14. Find the volume in the first octant bounded by the surfaces $x^2 + y^2 = az$, $x + 2y = 2a$. *Ans.* $\frac{5}{6}a^3$.

15. Find the volume in the first octant inside the cylinder $x^2 + y^2 = 2ax$ and outside the paraboloid $x^2 + y^2 = az$. *Ans.* $\frac{3}{4}\pi a^3$.

16. Find the volume bounded by the surfaces $4y^2 + 4z^2 = x^2$, $x = 4y$, $x = 2a$, $z = 0$. *Ans.* $\frac{1}{36}(4\pi - 3\sqrt{3})a^3$.

17. Find the volume in the first octant bounded by the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $\frac{1}{90}a^3$.

18. Find the centroid of the volume in the first octant bounded by the surfaces $az = x^2 + y^2$, $y = x$, $x = a$. *Ans.* $(\frac{4}{5}a, \frac{9}{20}a, \frac{7}{15}a)$.

19. Find the centroid of the volume in Ex. 1. *Ans.* $(\frac{199}{273}, \frac{6}{13}, \frac{158}{273})$.

20. Find the centroid of the volume in Ex. 4.

Ans. $(\frac{33}{50}a, \frac{11}{25}a, \frac{27}{50}a)$.

21. In Ex. 4, find the moment of inertia with respect to each coördinate plane and axis.

Ans. $I_{yz} = \frac{18}{25}Ma^2$, $I_{zx} = \frac{19}{70}Ma^2$, $I_{xy} = \frac{467}{1050}Ma^2$.

22. In Ex. 18, find the moment of inertia with respect to the z-axis.
Ans. $\frac{14}{15} Ma^2$.

23. In Ex. 6, find the moment of inertia with respect to the xy-plane.
Ans. $\frac{1}{12} Ma^2$.

24. Find the centroid of the volume bounded by the surfaces $z = 0$, $z = x$, $y = x$, $x^2 = ay$. *Ans.* $(\frac{3}{8}\, a, \frac{1}{2}\, a, \frac{3}{10}\, a)$.

25. Solve each part of Ex. 24 by a second method.

26. In Ex. 24, find the moment of inertia with respect to each coördinate plane and axis.
Ans. $I_{yz} = \frac{2}{5} Ma^2$, $I_{zx} = \frac{3}{10} Ma^2$, $I_{xy} = \frac{2}{15} Ma^2$.

27. Find the volume bounded by the surfaces $x^2 + z^2 = 3\,ay$, $x^2 + ay = a^2$. *Ans.* $\frac{3}{2}\pi a^3$.

169. The double integral in polar coördinates. Let S be a plane area bounded by a curve whose equation is given in polar coördinates. We may divide S into elements ΔS by means of concentric circles and radial lines, as in the figure. Then ΔS is the difference between two circular sectors of angle $\Delta\theta$ and radii r and $r + \Delta r$ respectively; *i.e.*

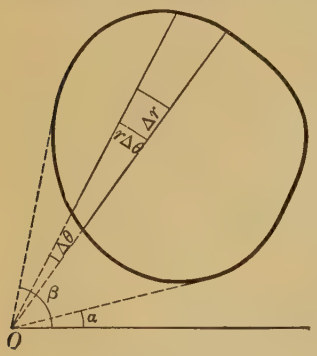


Fig. 92

$$\Delta S = \tfrac{1}{2}(r + \Delta r)^2\Delta\theta - \tfrac{1}{2} r^2\Delta\theta$$
$$= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}\overline{\Delta r}^2\Delta\theta.$$

Let $f(r, \theta)$ be a function of the polar coördinates defined at all points of the region S. If now we have a problem in hand in which the required quantity can be found approximately by the double summation $\sum\sum f(r, \theta)\Delta S$, argument similar to that of § 168 shows that this quantity can be expressed exactly as a double integral *:

$$(1) \qquad \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \sum\sum f(r,\ \theta) r \Delta r \Delta\theta = \int_{\alpha}^{\beta} \int_{r'}^{r''} f(r,\ \theta)\, r\, dr\, d\theta.$$

* By § 119, the infinitesimal of higher order $\frac{1}{2}\,\Delta r^2\,\Delta\theta$ can be neglected.

In particular, if the area S is the base of a solid bounded laterally by a vertical cylinder and cut off on top by the surface $z = f(r,\theta)$, the double integral (1) is obviously the volume of the solid.

Example: Find the volume in the first octant bounded by the surfaces $y = x$, $x = a$, $2az = x^2 + y^2$, using polar coördinates.

Divide the base into elements $r\,dr\,d\theta$ as in the figure, and erect on each elementary base a column of height z. Then

$$V = \int_0^{\frac{\pi}{4}} \int_0^{a \sec \theta} z\, r dr d\theta.$$

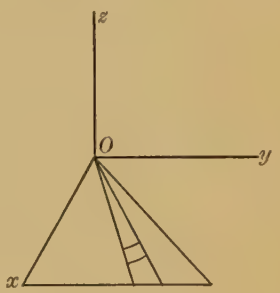


FIG. 93

But

$$x = r \cos \theta, \quad y = r \sin \theta,$$

so that the equation

$$2az = x^2 + y^2$$

becomes

$$2az = r^2.$$

Hence

$$V = \frac{1}{2a} \int_0^{\frac{\pi}{4}} \int_0^{a \sec \theta} r^3\, dr\, d\theta = \frac{1}{2a} \int_0^{\frac{\pi}{4}} \frac{r^4}{4} \Big]_0^{a \sec \theta} d\theta$$

$$= \frac{a^3}{8} \int_0^{\frac{\pi}{4}} \sec^4 \theta\, d\theta = \frac{a^3}{8} \left[\tan \theta + \frac{1}{3} \tan^3 \theta \right]_0^{\frac{\pi}{4}} = \frac{a^3}{6}.$$

EXERCISES

Solve the following exercises by use of polar coördinates. Where feasible, check by inverting the order of integration.

1. Find the volume above the xy-plane, inside the cylinder $x^2 + y^2 = a^2$ and below the paraboloid $x^2 + y^2 = az$. *Ans.* $\frac{1}{2}\pi a^3$.

2. Find the volume above the xy-plane common to the paraboloid $x^2 + y^2 + az = 4a^2$ and the cylinder $x^2 + y^2 = a^2$. *Ans.* $\frac{7}{2}\pi a^3$.

3. Find the volume inside the cylinder $x^2 + y^2 = a^2$ and outside the cone $x^2 + y^2 = z^2$. *Ans.* $\frac{4}{3}\pi a^3$.

4. Solve the example of § 120 by polar coördinates.

5. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x + y$. (Ex. 2, p. 282.) *Ans.* $\frac{2}{3}a^3$.

6. A round hole is bored through the center of a sphere. Find the volume cut out. *Ans.* $\frac{4}{3}\pi[a^3 - (a^2 - b^2)^{\frac{3}{2}}]$.

7. Find the volume inside the cylinder $x^2 + y^2 = 4a^2$ and outside the hyperboloid $x^2 + y^2 - z^2 = a^2$. *Ans.* $4\sqrt{3}\,\pi a^3$.

8. Find the volume inside the cylinder $x^2 + y^2 = a^2$ and outside the hyperboloid $z^2 - x^2 - y^2 = a^2$. *Ans.* $\frac{4}{3}\pi(2\sqrt{2} - 1)a^3$.

9. Find the volume in the first octant bounded by the surfaces $y = x$, $x = a$, $xy = az$. *Ans.* $\frac{1}{8}a^3$.

10. Find the volume in the first octant bounded by the surfaces $y = x$, $z = x$, $ay = x^2$. *Ans.* $\frac{1}{12}a^3$.

11. Solve Ex. 15, p. 283.

12. A square hole of side $2a$ whose axis is the z-axis is cut through the paraboloid of Ex. 2. Find the volume cut out. *Ans.* $\frac{40}{3}a^3$.

13. A vertical cylinder is passed through the circle $r = a\cos\theta$. Find the volume of the cylinder inside a sphere of radius a with center at the origin. *Ans.* $\frac{2}{3}(\pi - \frac{4}{3})a^3$.

14. Find the volume bounded by the surfaces $z = 0$, $x = 0$, $y = x$, $y = a$, $y^2 = a(z - x)$. *Ans.* $\frac{5}{12}a^3$.

15. In Ex. 1, find the centroid of that part of the volume that lies in the first octant. *Ans.* $\left(\frac{8a}{5\pi}, \frac{8a}{5\pi}, \frac{a}{3}\right)$.

16. In Ex. 1, find I_z and I_{xy} by integration; deduce the moments with respect to the other planes and axes.

Ans. $I_z = \frac{2}{3}Ma^2$; $I_{xy} = \frac{1}{6}Ma^2$.

17. In Ex. 3, find the centroid of the volume lying in the first octant.

Ans. $\left(\frac{3a}{2\pi}, \frac{3a}{2\pi}, \frac{3a}{8}\right)$.

18. In Ex. 3, find I_z and I_{xy}. *Ans.* $I_z = \frac{3}{5}Ma^2$; $I_{xy} = \frac{1}{5}Ma^2$.

19. Find the centroid of the volume in the first octant bounded by the surfaces $y = x$, $x = a$, $zx = ay$. *Ans.* $(\frac{2}{3}a, \frac{4}{9}a, \frac{1}{3}a)$.

20. Find the moment of inertia of the volume in Ex. 19 with respect to each coördinate plane. *Ans.* $I_{yz} = \frac{1}{2}Ma^2$; $I_{zx} = \frac{1}{4}Ma^2$; $I_{xy} = \frac{1}{6}Ma^2$.

21. Find the centroid of half of a cone of revolution.

Ans. $\left(\frac{a}{\pi}, 0, \frac{h}{4}\right)$.

170. Solids of revolution: polar coördinates. Imagine a solid of revolution to be generated by rotating the area of Fig. 92, p. 284, about Ox. The elementary area ΔS evidently generates a slender circular filament of cross-section $r\,\Delta r\,\Delta\theta$ and circumference $2\,\pi r \sin\theta$, whence the volume of the filament (apart from infinitesimals of higher order) is * $2\,\pi r^2 \sin\theta\,\Delta r\,\Delta\theta$. Hence, by the usual reasoning,

$$V = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \sum 2\,\pi r^2 \sin\theta\,\Delta r\,\Delta\theta = 2\,\pi \int_{\alpha}^{\beta} \int_{r'}^{r''} r^2 \sin\theta\,dr\,d\theta.$$

EXERCISES

Solve the following by the method of § 170.

1. Find the volume of a sphere.

2. Find the volume cut from a sphere by one sheet of a cone of vertical angle $2\,\alpha$ with its vertex at the center of the sphere. Check by inverting the order of integration. *Ans.* $\frac{2}{3}\,\pi a^3(1 - \cos\alpha)$.

3. The curve $r^2 = a^2 \sin\theta$ revolves about the y-axis. Find the volume generated; check by inversion of order. *Ans.* $\frac{8}{15}\,\pi a^3$.

4. Find the volume generated by revolving the cardioid $r = a(1 - \sin\theta)$ about its line of symmetry. Integrate in both orders. *Ans.* $\frac{8}{3}\,\pi a^3$.

5. Find the volume of a paraboloid of revolution bounded by a right section through the focus, taking the equation of the generating parabola in the form $r = \dfrac{2\,a}{1 - \cos\theta}$. *Ans.* $2\,\pi a^3$.

6. Find the volume generated by revolving a circle about one of its tangents. Take the center of the circle at O. *Ans.* $2\,\pi^2 a^3$.

7. Solve Ex. 6, using the circle $r = 2\,a \cos\theta$.

8. Find the volume generated by revolving about Ox the area bounded by the curves $y = x$, $y = 2\,x$, $ay = x^2$. *Ans.* $\frac{64}{15}\,\pi a^3$.

9. Find the volume generated by revolving the area of Ex. 8 about Oy. *Ans.* $\frac{5}{2}\,\pi a^3$.

10. Find the volume of the prolate spheroid generated by revolving about its major axis the ellipse

$$r = \frac{l}{1 - e\cos\theta},$$

where e is the eccentricity. *Ans.* $\dfrac{4\,\pi l^3}{3(1 - e^2)^2}$.

* A strict proof of this statement is easily obtained by the second proposition of Pappus (Ex. 30, p. 211).

11. Find the volume generated by revolving one loop of the curve $r = a \sin 2\theta$ about the initial line. *Ans.* $\frac{32}{105}\pi a^3$.

12. Find the centroid of a hemisphere. *Ans.* $\bar{x} = \frac{3}{8}a$.

13. Find the moment of inertia of a sphere with respect to a diametral plane. *Ans.* $\frac{1}{5}Ma^2$.

14. Find the centroid of the volume in Ex. 2.

Ans. $\bar{x} = \frac{3}{8}a(1 + \cos\alpha)$.

15. Find the centroid of the volume in Ex. 8. *Ans.* $\bar{x} = \frac{315}{488}a$.

16. Find the moment of inertia of the volume in Ex. 9, with respect to the y-axis. *Ans.* $I_y = \frac{42}{25}Ma^2$.

17. Find the moment of inertia of the volume in Ex. 9, with respect to the zx-plane. *Ans.* $\frac{51}{10}Ma^2$.

171. Evaluation by inversion of order. We have seen that inversion of the order of integration frequently affords a useful check on the value of a double integral. Furthermore, it may happen that an integral given, for example, in the form (2) of § 168 is difficult or impossible to evaluate, but that when changed to the equivalent form (3), it becomes simple. An instance of this is furnished by the following

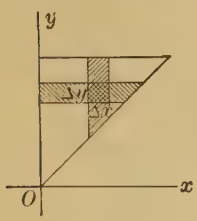


FIG. 94

Example: Evaluate $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$.

This integral cannot be evaluated directly, since the function e^{y^2} is not integrable in terms of elementary functions. But a study of the limits shows that the field of integration is the triangle bounded by the lines $x = 0$, $y = x$, $y = 1$. Hence

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

$$= \int_0^1 \Big[e^{y^2}x\Big]_0^y dy$$

$$= \int_0^1 e^{y^2}y\, dy = \tfrac{1}{2}\, e^{y^2}\Big]_0^1$$

$$= \tfrac{1}{2}(e - 1).$$

172. Evaluation by change of coördinate system. An integration that is impossible in cartesian coördinates can sometimes be performed after a change to polar coördinates, or vice versa. The process is illustrated by the following

Example: Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

Here we can readily perform the first integration but not the second, and inversion of order is useless. The field of integration is evidently the first quadrant of the circle of radius a with center at O. Replacing $\sqrt{x^2+y^2}$ by r, and $dy\,dx$ by the polar element $r\,dr\,d\theta$, we have *

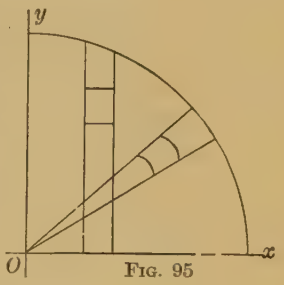


FIG. 95

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^a r^2\, dr\, d\theta = \tfrac{1}{6}\pi a^3$$

EXERCISES

Evaluate the following integrals.

1. $\int_0^1 \int_y^1 \frac{y\,dx\,dy}{\sqrt{x^2+y^2}}$. *Ans.* $\frac{1}{2}(\sqrt{2}-1)$.
2. $\int_0^1 \int_{2y}^2 y\sqrt{x^2+y^2}\, dx\, dy$. *Ans.* $\frac{8}{15}\sqrt{5}-\frac{4}{3}$.
3. $\int_0^1 \int_x^1 x \sin y^3\, dy\, dx$. *Ans.* 0.077.
4. $\int_0^2 \int_{\frac{1}{2}y}^1 \frac{dx\,dy}{(1+x^2)^3}$. *Ans.* $\frac{3}{8}$.
5. $\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dx\, dy$. *Ans.* 0.159.
6. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3}\, dy\, dx$. *Ans.* 0.406.
7. $\int_{-1}^0 \int_{-x}^1 \frac{x^2\,dy\,dx}{1+y^4}$. *Ans.* 0.058.
8. $\int_0^4 \int_{\sqrt{y}}^2 \frac{y\,dx\,dy}{\sqrt{1+x^5}}$. *Ans.* 0.949.

* Cf. Ex. 3, p. 285.

Evaluate the following integrals by transforming to polar coördinates.

9. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx.$ *Ans.* 1.35.

10. $\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(x^2 + y^2)\, dy\, dx.$ *Ans.* 0.661.

11. $\int_0^1 \int_y^1 \frac{y\, dx\, dy}{\sqrt{x^2 + y^2}}.$ (Ex. 1.)

12. $\int_0^1 \int_{2y}^2 y\sqrt{x^2 + y^2}\, dx\, dy.$ (Ex. 2.)

13. $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx.$ (§ 171.)

14. $\int_0^1 \int_x^1 x \sin y^3\, dy\, dx.$ (Ex. 3.)

15. $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2\sqrt{x^2 + y^2}\, dy\, dx.$ *Ans.* $\frac{1}{20}\pi a^5.$

16. $\int_0^a \int_0^x \frac{x^2\, dy\, dx}{\sqrt{x^2 + y^2}}.$ *Ans.* $\frac{a^3}{3}\log(1 + \sqrt{2}).$

17. Find the area in the first quadrant under the curve $y = e^{-\frac{1}{2}x^2}$ by noting that

$$\left(\int_0^\infty e^{-\frac{1}{2}x^2}\, dx\right)^2 = \left(\int_0^\infty e^{-\frac{1}{2}x^2}\, dx\right)\cdot\left(\int_0^\infty e^{-\frac{1}{2}y^2}\, dy\right) = \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(x^2+y^2)} dy\, dx.$$

Ans. $\frac{1}{2}\sqrt{2\pi}.$

18. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\sec\theta} \frac{r\, dr\, d\theta}{1 + r^2\sin^2\theta}$ by transforming to cartesian coördinates. *Ans.* $\frac{1}{2}\pi.$

173. Heterogeneous masses. The density of a homogeneous mass has been defined in § 125 as the ratio of the mass to the volume it occupies:

$$\delta = \frac{M}{V}.$$

For a *heterogeneous* mass, *i.e.* one whose density varies from point to point, we must introduce the idea of *density at a point.*

Consider an element of volume ΔV including a point P, and let ΔM denote the mass contained in ΔV. Then the

ratio $\frac{\Delta M}{\Delta V}$ is the *average* density of ΔV. If ΔV approaches 0 in such a way that P is always included, the ratio $\frac{\Delta M}{\Delta V}$ in general approaches a limit * δ, called the *density at the point P:*

$$\delta = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV}.$$

For a mass distributed over a surface S, the idea of "surface density" must be introduced:

$$\delta = \lim_{\Delta S \to 0} \frac{\Delta M}{\Delta S} = \frac{dM}{dS}.$$

Similarly, for a mass distributed along a curve C, the "linear density" is

$$\delta = \lim_{\Delta s \to 0} \frac{\Delta M}{\Delta s} = \frac{dM}{ds}.$$

The mass of a heterogeneous body whose density at any point is given as a function of the coördinates of the point can be found by integration. We have only to choose a suitable mass-element and integrate over the whole body:

$$M = \int \delta \, dV,$$

etc. The essential point to be noted is that *the element itself must be homogeneous,*† since otherwise the mass of the element cannot be computed and hence the integral cannot be built up.

Examples: (*a*) Find the mass of a circular cone whose density varies as § the distance from the axis.

* In the most general case this is true only if ΔV is infinitesimal of the third order, as in § 174.

† By this is meant that the density at different points of the element varies only by *infinitesimal* amounts; cf. the examples below.

§ To say that *a varies as b*, or *a is proportional to b*, means that $a = kb$, where k is constant.

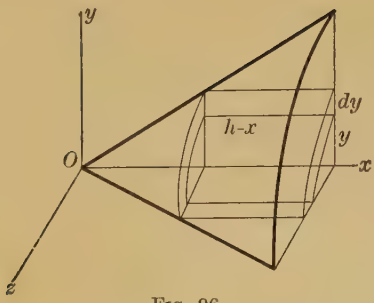


FIG. 96

Let us take the vertex of the cone at the origin and its axis along Ox. If we divide the mass into cylindrical shells about the axis, each element will be "homogeneous" of density

$$\delta = ky.$$

We have

$$dV = 2\pi y(h - x)dy,$$

$$M = \int \delta\, dV = 2\pi k \int_0^a y^2(h - x)\, dy$$

$$= 2\pi k \int_0^a y^2 \left(h - \frac{h}{a}y\right) dy = \frac{1}{6}\pi k a^3 h.$$

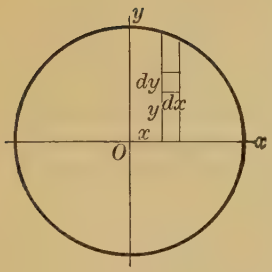


FIG. 97

(*b*) Find the mass of a circular plate if the density is proportional to the sum of the distances from two perpendicular diameters.

Taking the two diameters as axes, we have

$$\delta = k(x + y),$$

whence

$$M = 4k \int_0^a \int_0^{\sqrt{a^2 - x^2}} (x + y)\, dy\, dx, \text{ etc.}$$

EXERCISES

Determine the following masses.

1. A straight rod whose density is proportional to the distance from one end. *Ans.* $\frac{1}{2}kl^2$.

2. A semicircular wire whose density varies as the distance from the diameter joining the ends. *Ans.* $2ka^2$.

3. A circular plate whose density varies as the distance from a fixed diameter. *Ans.* $\frac{4}{3}ka^3$.

4. A circular plate whose density varies as the distance from the center. *Ans.* $\frac{2}{3}\pi ka^3$.

5. A circular plate whose density varies inversely as the distance from the center. *Ans.* $2\pi ka$.

6. A rectangular plate whose density is proportional to the square of the distance from one side. *Ans.* $\frac{1}{3}kab^3$.

7. A square plate whose density is proportional to the distance from a diagonal. *Ans.* $\frac{1}{6}\sqrt{2}\,ka^3$.

8. An isosceles right triangle whose density is proportional to the square of the distance from the hypotenuse. *Ans.* $\frac{1}{24}ka^4$.

9. A circular ring (area bounded by two concentric circles) whose density is inversely proportional to the square of the distance from the center. *Ans.* $2\pi k \log \frac{b}{a}$.

10. A circular cylinder whose density is proportional to the distance from the axis. *Ans.* $\frac{2}{3}\pi ka^3h$.

11. A circular cylinder whose density is proportional to the square of the distance from the base. *Ans.* $\frac{1}{3}\pi ka^2h^3$.

12. A sphere whose density varies as the distance from a fixed diametral plane. *Ans.* $\frac{1}{2}\pi ka^4$.

13. A sphere whose density varies inversely as the distance from a fixed diameter. *Ans.* π^2ka^2.

14. A sphere whose density varies inversely as the square of the distance from the center. *Ans.* $4\pi ka$.

15. An ellipsoid of revolution whose density is proportional to the square of the distance from the axis. *Ans.* $\frac{8}{15}\pi kab^4$.

16. A tetrahedron with three mutually perpendicular faces, the density varying as the distance from one of the faces. *Ans.* $\frac{1}{24}ka^2bc$.

17. A circular cone whose density is proportional to the square of the distance from the base. *Ans.* $\frac{1}{30}\pi ka^2h^3$.

18. A spherical surface whose density varies as the distance from a fixed diameter. *Ans.* π^2ka^3.

19. A spherical surface whose density varies as the distance from a diametral plane. *Ans.* $2\pi ka^3$.

20. In example (*a*) of § 173, find the shape of the heaviest cone having a given slant height. *Ans.* $a = \sqrt{3}\,h$.

21. Solve Ex. 20 by trigonometric functions. *Ans.* $\alpha = 60°$.

22. In Ex. 17, find the shape of the heaviest cone if the slant height is given. *Ans.* $h = \frac{1}{2}\sqrt{6}\,a$.

23. Solve Ex. 22 by trigonometric functions. *Ans.* $\alpha = 39°\ 14'$.

24. In Ex. 10, find the moment of inertia with respect to an axial plane. *Ans.* $\frac{3}{10}Ma^2$.

25. In Ex. 10, find the moment of inertia with respect to the base.

26. In Ex. 11, find the moment of inertia with respect to (*a*) the axis; (*b*) the base. *Ans.* (*b*) $\frac{3}{5} Mh^2$.

27. In Ex. 17, find the centroid. *Ans.* $\bar{x} = \frac{1}{2} h$.

Determine the following masses.

28. A rectangular plate whose density is proportional to the sum of the distances from two adjacent sides. *Ans.* $\frac{1}{2} kab(a + b)$.

29. The mass in example (*b*) above. *Ans.* $\frac{8}{3} ka^3$.

30. The mass in example (*b*), using polar coördinates.

31. A circular plate whose density varies as the distance from a point on the circumference. *Ans.* $\frac{32}{9} ka^3$.

32. A square whose density is proportional to the distance from one corner. *Ans.* $0.765\, ka^3$.

33. A circular cylinder whose density varies as the square of the distance from the center of the base. *Ans.* $\frac{1}{6} \pi ka^2 h(3a^2 + 2h^2)$.

34. A sphere whose density varies as the distance from a point on the surface. (See § 170.) *Ans.* $\frac{8}{5} \pi ka^4$.

35. A circular cone whose density varies inversely as the distance from the vertex. *Ans.* $\pi kh(\sqrt{a^2 + h^2} - h)$.

36. Find the centroid of one half of the mass in Ex. 10.

Ans. $\bar{y} = \dfrac{3a}{2\pi}$.

37. Find the centroid of one half of the mass in Ex. 14.

Ans. $\bar{x} = \frac{1}{4} a$.

38. In Ex. 31, find the centroid. *Ans.* $\bar{x} = \frac{8}{5} a$.

39. In Ex. 34, find the centroid. *Ans.* $\bar{x} = \frac{8}{7} a$.

174. Triple integrals. We have seen that the integral of a function of one variable, extended over a given interval, may be interpreted as the area under a plane curve. Again, the integral of a function of two variables extended over a plane region may be interpreted as the volume under a surface. If now we have a function of three variables defined at all points of a portion space, no similar geometric interpretation for the integral of the function throughout the given region is possible, since geometric intuition fails in space of four dimensions. Nevertheless the meaning of such an integral may be made plain by analogy with the earlier cases.

Suppose we have given a function $f(x, y, z)$ defined at all points of a three-dimensional region V. Let us pass through V three sets of planes parallel to the coördinate planes, thus dividing V into elementary rectangular parallelepipeds of volume $\Delta x \Delta y \Delta z$, together with smaller

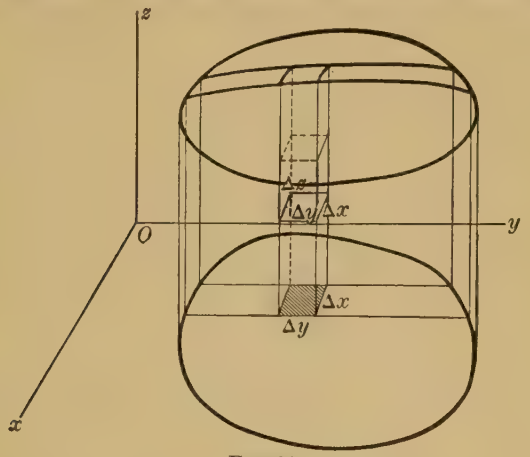


FIG. 98

irregular portions around the boundary. Now multiply the volume of each element by the value of the function at some point within the element, say at its center, and form the sum of these products. The triple limit

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum f(x, y, z)\, \Delta x\ \Delta y\ \Delta z$$

is defined as the value of the *triple integral* of $f(x, y, z)$ throughout the region V.

This limit may be evaluated by three successive integrations (cf. § 168):

$$T = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum f(x, y, z)\, \Delta x\ \Delta y\ \Delta z$$

$$= \int_a^b \int_{y'}^{y''} \int_{z'}^{z''} f(x, y, z)\, dz\, dy\, dx.$$

The first integration extends over a vertical column of base $\Delta y\ \Delta x$; the limits z', z'' are the extreme values of z in this column, and are in general functions of both x and y. The integration with respect to y is extended over a slice parallel to the yz-plane; the limits y' and y'' are the extreme values of y in this slice, and are functions of x alone. In the final integration the limits are of course the extreme values of x in the whole region.

While triple integrals are of great importance in more advanced theory, they find comparatively little application in elementary work, single or double integrals being sufficient in most problems.

It is hardly necessary to say that such transformations as inversion of order and change from one coördinate system to another are allowable and useful with triple integrals, just as with double integrals.

Example: Find the mass of a sphere whose density is proportional to the sum of the distances from three mutually perpendicular diametral planes.

In cartesian coördinates we have

$$M = 8k\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (x + y + z)\, dz\, dy\, dx.$$

In cylindrical coördinates (polar coördinates r, θ with the cartesian z) the integral is

$$M = 8k\int_0^{\frac{\pi}{2}} \int_0^a \int_0^{\sqrt{a^2-r^2}} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta.$$

EXERCISES

1. Find the mass of a cube whose density varies as the sum of the distances from three adjacent faces. *Ans.* $\frac{3}{2}ka^4$.

2. In Ex. 1, find the centroid. *Ans.* $(\frac{5}{9}a, \frac{5}{9}a, \frac{5}{9}a)$.

3. In Ex. 1, find the moment of inertia with respect to one of the faces mentioned. *Ans.* $\frac{7}{18}Ma^2$.

4. Find the mass of a cube whose density varies as the square of the distance from one corner. *Ans.* ka^5.

5. In Ex. 4, find the centroid. *Ans.* $(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a)$.

6. Find the mass of the tetrahedron bounded by the coördinate planes and the plane $x + y + z = a$, if the density is proportional to the sum of the distances from the coördinate planes. *Ans.* $\frac{1}{8}ka^4$.

7. In Ex. 6, find the centroid. *Ans.* $(\frac{4}{15}a, \frac{4}{15}a, \frac{4}{15}a)$.

8. In Ex. 6, find the moment of inertia with respect to one of the coördinate axes. *Ans.* $\frac{3}{5}Ma^2$.

CHAPTER XXV

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

175. General definition. A *differential equation* is an equation that involves derivatives or differentials. Various examples have arisen in our previous work, of which the following may be mentioned:

(1) $\dfrac{dx}{dt} = -\dfrac{1}{(t+1)^2}$, (p. 16)

(2) $y'' = -\dfrac{a^2}{y^3}$, (p. 31)

(3) $3x^2\,dx + 3y^2\,dy - 3ax\,dy - 3ay\,dx = 0$, (p. 90)

(4) $\dfrac{d^2x}{dt^2} = -k^2x$, (p. 54)

(5) $dy = 2\cos 2\theta\,d\theta$, (p. 90)

(6) $\dfrac{d^2x}{dt^2} = 2e^{-t}$, (p. 141)

(7) $y^{(n)} = a^n e^{ax}$, (p. 80)

(8) $\dfrac{\partial z}{\partial x} = \dfrac{-y}{x^2+y^2}$, (p. 268)

(9) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$. (p. 268)

Equations containing partial derivatives, such as examples (8) and (9), are called *partial differential equations.* A study of these equations is beyond the limits of this book.

176. Order of a differential equation. The *order* of a differential equation is the order of the highest derivative that occurs in it. Thus, in § 175, examples (1), (3), (5), are of the first order, (2), (4), (6) are of the second order, (7) is of the n-th order.

In the applications, equations of the first and second orders are of predominant importance, and we shall be chiefly concerned with these two types.

177. Solutions of a differential equation. A *solution* of a differential equation is any relation between the variables by virtue of which the differential equation is satisfied. Thus equation (1) of § 175 is true if

$$x = \frac{1}{t+1} + c,$$

where c is arbitrary; hence this relation is a solution of the equation. A solution of (6) is easily seen to be

$$x = 2\,e^{-t} + c_1 t + c_2.$$

It appears from these examples that a solution of a differential equation may involve one or more arbitrary constants; we shall find this to be true in general. It follows that each equation has an infinity of solutions, obtained by assigning different values to the arbitrary constants.

By analogy with the integral calculus, a solution of a differential equation is often called an *integral* of the equation, and the arbitrary constants are called *constants of integration.*

In the remainder of this chapter we shall consider only equations of the first order.

178. The general solution. Suppose there is given a relation (free of derivatives) between x, y, and an arbitrary constant:

(1) $$F(x, y, c) = 0.$$

Geometrically this equation represents a *family of curves,* whose individual members are obtained by assigning particular values to c.

If we differentiate (1) with respect to x, the arbitrary constant c may be eliminated from the equation thus formed and the original equation. The result of this elimination is evidently an equation involving x, y, and y'; *i.e.* it is a differential equation of the first order:

(2) $$\Phi(x, y, y') = 0.$$

As this equation does not contain c, it represents a property common to *all the curves* of the above-mentioned family.

Since equation (2) is true by virtue of equation (1), it follows that (1) is a solution of (2). The family of curves (1) are called the *integral curves* corresponding to (2).

If a solution of a differential equation of the first order contains an arbitrary constant, it is called the *general solution:* hence (1) is the general solution of (2). It can be shown that, in general, corresponding to every differential equation of the form (2) there exists a general solution (1); methods of finding this solution in various cases will be considered presently.

It may be worth while to point out that, if the differential equation has the simple form

$$\frac{dy}{dx} = f(x),$$

the integral calculus gives us the general solution at once:

$$y = \int f(x)\, dx + c.$$

It should also be noted that while in the integral calculus the constant of integration always appears as an additive constant, this is not true in general in the solution of a differential equation; the constant often enters in other ways.

Examples: (*a*) Find the differential equation whose general solution is

$$y = ce^{2x}.$$

Differentiating, we find

$$dy = 2\, ce^{2x} dx;$$

eliminating c by division, we get

$$\frac{dy}{y} = 2\, dx.$$

This example illustrates the fact that the arbitrary constant is not always additive.

(*b*) Find by inspection the general solution of the equation

$$xdy + ydx = 0.$$

The answer is seen at once to be

$$xy = c.$$

179. Particular solutions. A solution obtained from the general solution by assigning a particular value to the arbitrary constant is called a *particular solution* of the differential equation. Thus in example (*b*), § 178, the equations $xy = 0$, $xy = 5$, etc., are particular solutions.

In applied problems involving differential equations we are in most cases concerned with a particular solution. Nevertheless the determination of the general solution is usually a necessary preliminary step, after which the required particular solution is found by determining the arbitrary constant from given *initial conditions*. The process is illustrated by the examples of § 84, which should be reviewed at this point.

Differential equations involving y' to a degree higher than the first may in some cases have a so-called *singular solution*, which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. As such solutions are of little importance in most of the elementary applications, we shall omit a discussion of them.

EXERCISES

In the following cases, find the differential equation whose general solution is the given equation.

1. $y = 2x^3 + x + c$. *Ans.* $y' = 6x^2 + 1$.
2. $y = \sin^2 2x + c$. *Ans.* $y' = 4 \sin 2x \cos 2x$.
3. $y = cx$. *Ans.* $x\,dy = y\,dx$.
4. $y = ce^x$. *Ans.* $dy = y\,dx$.
5. $y = cx + c$. *Ans.* $(x + 1)\,dy = y\,dx$.
6. $x = ct + c^2$. *Ans.* $x = t\dfrac{dx}{dt} + \left(\dfrac{dx}{dt}\right)^2$.
7. $xy + cy = 1$. *Ans.* $y^2\,dx + dy = 0$.
8. $x^2 = 2cy + c^2$. *Ans.* $xy'^2 = 2yy' + x$.
9. $y = \log cx$ *Ans.* $x\,dy = dx$.
10. $c(x + y) = xy$. *Ans.* $x^2\,dy + y^2\,dx = 0$.

Find the differential equation of each of the following families of curves.

11. Straight lines parallel to Ox. *Ans.* $y' = 0$.

12. Straight lines making an angle of 60° with Ox.

13. Straight lines through the origin.

14. Straight lines through the point (1, 3).

Ans. $(x - 1)\,dy = (y - 3)\,dx$.

15. Straight lines making an angle of 45° with the line $y = 3x + 2$.

Ans. $y' = \frac{1}{2}$, or $y' = -2$.

16. Circles with center at $(1, -2)$.

Ans. $(x - 1)\,dx + (y + 2)\,dy = 0$.

17. Circles through the origin with center on Ox.

Ans. $2xy\,dy = (y^2 - x^2)\,dx$.

18. Parabolas with vertex at (0, 0) and axis coinciding with Ox.

Ans. $2x\,dy = y\,dx$.

19. Parabolas with focus at the origin and axis coinciding with the x-axis. *Ans.* $yy'^2 + 2xy' = y$.

20. Ellipses having the points $(\pm 3, 0)$ as ends of one axis.

Ans. $xy\,dx + (9 - x^2)\,dy = 0$.

21. Hyperbolas having the coördinate axes as asymptotes.

Ans. $x\,dy + y\,dx = 0$.

22. Hyperbolas having the lines $x - y = 3$, $2x + y + 1 = 0$ as asymptotes. *Ans.* $(4x - y - 5)\,dx = (x + 2y + 4)\,dy$.

180. Separation of variables. In the remainder of this chapter we show how to find the general solution of a differential equation of the first order in some of the more important cases.

Every differential equation of the first order, and of the first degree in y', can evidently be written in the form

$$M\,dx + N\,dy = 0,$$

where in general M and N are functions of both x and y. It is often possible to transform the equation so that M is a function of x alone and N is a function of y alone; this transformation is called *separation of variables*. When the variables have been separated, the differential equation may be solved by a simple integration, as in the following

Example: Solve the equation

$$xy\,dx + (x^2 + 1)\,dy = 0.$$

After division by $y(x^2 + 1)$ the equation takes the form

$$\frac{x\,dx}{x^2 + 1} + \frac{dy}{y} = 0.$$

Integrating, we get

$$\tfrac{1}{2}\log(x^2 + 1) + \log y = c,$$

or

$$\log y\sqrt{x^2 + 1} = c,$$
$$y\sqrt{x^2 + 1} = e^c,$$
$$y^2(x^2 + 1) = c',$$

where $c' = e^{2c}$.

EXERCISES

Solve the following differential equations.

1. $\dfrac{dv}{dt} = -kv^2$. *Ans.* $kvt + cv = 1$.
2. $dy = 2xy\,dx$. *Ans.* $y = ce^{x^2}$.
3. $x\,dy + 2y\,dx = 0$. *Ans.* $x^2y = c$.
4. $\dfrac{dx}{dt} = 1 - \sin 2t$. *Ans.* $x = t + \tfrac{1}{2}\cos 2t + c$.
5. $(1 - x)\,dy = (1 + y)\,dx$. *Ans.* $xy + x - y = c$.
6. $\dfrac{dx}{dt} = tx^2$. *Ans.* $t^2x + cx + 2 = 0$.
7. $\dfrac{dy}{dx} + y^2 = a^2$. *Ans.* $\dfrac{y + a}{y - a} = ce^{2ax}$.
8. $x^3\,dy - xy^2\,dx = 2ay^2\,dx$. *Ans.* $x^2y = c(x^2 - xy - ay)$.
9. $y\,dx + x\,dy = xy(dy - dx)$. *Ans.* $\log xy + x - y = c$.
10. $\sin x \cos y\,dx = \cos x \sin y\,dy$. *Ans.* $\cos y = c\cos x$.
11. $y\,dx = xy\,dx + x^2\,dy$. *Ans.* $x\log xy + 1 = cx$.
12. $y^2\,dx + y\,dy + x^2y\,dy = dx$. *Ans.* $\arctan x + \tfrac{1}{2}\log(1 - y^2) = c$.
13. $x^2\,dx - y\,dy + x^2y^2\,dx + x^3y\,dy = 0$. *Ans.* $(x^3 - 1)^2(y^2 + 1)^3 = c$.
14. $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$. *Ans.* $x + y = c(1 - xy)$.
15. $\sqrt{1 - y^2}\,dx + \sqrt{1 - x^2}\,dy = 0$.
Ans. $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$.

In the following cases, find a particular solution satisfying the given condition.

16. $x^2\,dy + y^2\,dx = 0$; $y = 3$ when $x = 2$. *Ans.* $5xy = 6x + 6y$.

17. $dx = (6t + 2)dt$; $x = -1$ when $t = 0$. *Ans.* $x = 3t^2 + 2t - 1$.

18. $y' = x^2y^2$; $y = 3$ when $x = 1$. *Ans.* $x^3y - 2y + 3 = 0$.

19. $\dfrac{dy}{dx} = -\dfrac{x-1}{y}$; $y = 4$ when $x = 3$. *Ans.* $(x-1)^2 + y^2 = 20$.

20. $xy\,dx + x^2\,dy + y\,dx = 0$; $y = 2$ when $x = 1$.

Ans. $\log xy = \dfrac{1}{x} - 0.307$.

21. $\dfrac{dy}{dx} = -y\tan x$; $y = 0.421$ when $x = 10°\ 30'$.

Ans. $y = 0.428\cos x$.

22. $dy - dx = y^2\,dx$; $y = -0.563$ when $x = 0$.

Ans. $y = \tan(x - 0.513)$.

23. $\sqrt{1-x^2}\,d\theta = dx$; $x = 0.242$ when $\theta = 8°20'$.

Ans. $x = \sin(\theta + 0.099)$.

24. Find the equation of the family of curves whose slope at any point equals the ordinate of the point. Determine that curve of the family that passes through (0, 1). *Ans.* $y = e^x$.

25. Find a function whose first derivative is equal to the square of the original function. Interpret geometrically. *Ans.* $y = \dfrac{-1}{x + c}$.

26. Determine the family of curves whose slope at any point is equal to the product of the coördinates of the point. Find the curve of this family that passes through the point (0, 1), and trace it.

Ans. $y = e^{\frac{1}{2}x^2}$.

27. Find the curve through $(a, 2a)$, having the property $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

Ans. $xy = 2a^2$.

28. Find the curve through (1, 6), having the property $\dfrac{dy}{dx} = \dfrac{x}{y}$.

Ans. $y^2 = x^2 + 35$.

29. Find the curve through $(4, -1)$, having the property $\dfrac{dy}{dx} = -\dfrac{3x}{4y}$. *Ans.* $3x^2 + 4y^2 = 52$.

30. Determine the family of curves represented by the equation $y' = e^y$. *Ans.* $x + e^{-y} = c$.

31. In Ex. 30, find the curve (*a*) that passes through (0, 0); (*b*) that crosses the line $x = 1$ at an angle of 45°. Trace these curves.

32. Find the curve through $(a, 0)$ such that $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$.

Ans. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

181. Coefficients homogeneous of the same degree. A polynomial in x and y is said to be *homogeneous* if all the terms are of the same degree in x and y. More generally, any function of x and y is said to be *homogeneous of the n-th degree* if, when x and y are replaced by kx and ky the result is the original function multiplied by k^n. Thus the function

$$x + \sqrt{x^2 - y^2} + y \log \frac{y}{x}$$

is homogeneous of the first degree.

If, in the equation

$$M\,dx + N\,dy = 0,$$

the coefficients M and N are homogeneous functions *of the same degree*, it is easily seen that the equation when solved for y' takes the form

$$y' = f\left(\frac{y}{x}\right):$$

i.e. y' is a function of $\frac{y}{x}$ alone. This suggests the substitution of a new variable v for the ratio $\frac{y}{x}$, *i.e.* the substitution

(1) $$\boldsymbol{y = vx, \quad dy = v\,dx + x\,dv.}$$

This substitution always produces a differential equation in v and x in which *the variables are separable.*

Example: Solve the equation

$$(x + y)dx - x\,dy = 0.$$

Substituting for y and dy by (1),

we find $$(x + vx)dx - x(v\,dx + x\,dv) = 0$$

or

$$dx - x\,dv = 0.$$

The variables can now be separated:

$$\frac{dx}{x} - dv = 0,$$

$$\log x - v + c = 0,$$

or, since

$$v = \frac{y}{x},$$

$$y = x \log x + cx.$$

EXERCISES

Solve the following differential equations.

1. $(x + y)y' + x - y = 0$. *Ans.* $\arctan \frac{y}{x} + \frac{1}{2}\log (x^2 + y^2) = c$.

2. $(x^2 + y^2)dx - 2xy\,dy = 0$. *Ans.* $x^2 - y^2 = cx$.

3. $(xy - x^2)\frac{dy}{dx} = y^2$. *Ans.* $y = ce^{\frac{y}{x}}$.

4. $x^3\,dy + y^3\,dx = 0$. *Ans.* $x^2y^2 = c^2(x^2 + y^2)$.

5. $2y\,dx + 3x\,dy = 0$. *Ans.* $x^2y^3 = c$.

6. $u\,dv - v\,du - \sqrt{u^2 + v^2}\,du = 0$. *Ans.* $u^2 = c^2 + 2cv$.

7. $y\,dx + \sqrt{1 - x^2}\,dy = 0$.

8. $2uv\,du + (v^2 - 3u^2)dv = 0$. *Ans.* $v^3 = c(u^2 - v^2)$.

9. $(x - 2y)dx + y\,dy = 0$. *Ans.* $\log (x - y) + \frac{x}{x - y} = c$.

10. $(x^2 + 2y^2)dx = xy\,dy$. *Ans.* $x^4 = c^2(x^2 + y^2)$.

11. $(x^3 - y^3)dx + xy^2\,dy = 0$. *Ans.* $x^3e^{\frac{y^3}{x^3}} = c$.

12. $(x^2 - xy + y^2)dx + x^2\,dy = 0$. *Ans.* $\log x + \arctan \frac{y}{x} = c$.

13. $(x^2 + y^2)dy + xy\,dx = 0$. *Ans.* $y^2(2x^2 + y^2) = c$.

14. Show that, if M and N are homogeneous of the same degree, the equation

$$M\,dx + N\,dy = 0$$

can always be put in the form

$$y' = f\left(\frac{y}{x}\right).$$

15. Give a general proof of the fact that, in the problem of § 181, the substitution $y = vx$ always leads to an equation in which the variables are separable.

182. Exact differential equations. Given a differential equation

$$M\,dx + N\,dy = 0,$$

it may happen that the entire left member is exactly the differential of some function of x and y, say $F(x, y)$. In this case the solution can be written down at once: it is of course

$$F(x, y) = c.$$

An equation of this type is called an *exact differential equation.*

Example: Solve the equation

$$3\,x^2y\,dx + x^3\,dy = x\,dy + y\,dx.$$

The solution is

$$x^3y = xy + c.$$

183. Integrating factors. If an equation is not exact, it may be possible to find some quantity, in general involving both x and y, such that when both members are multiplied by this quantity the equation becomes exact. The factor thus introduced is called an *integrating factor.*

It may be pointed out that an equation becomes exact after separation of the variables, and the factor which separates the variables is an integrating factor. Thus in the example of § 180 the quantity $\frac{1}{y(x^2+1)}$ is an integrating factor. This, however, is only one instance of the use of an integrating factor; the method succeeds in many cases where the variables cannot be separated.

Examples: (*a*) Solve the equation

$$3\,y\,dx + x\,dy = dx.$$

The sum occurring in the left member suggests the differential of a product; the coefficient 3 suggests that one of the factors of this product is a cube. It is not hard to see that x^2 is an integrating factor:

$$3\,x^2y\,dx + x^3\,dy = x^2\,dx,$$
$$x^3y = \tfrac{1}{3}\,x^3 + c.$$

(*b*) Solve the equation

$$x\,dy - y\,dx = x^2\,dx.$$

If we note that the differential of $\frac{y}{x}$ is $\frac{x\,dy - y\,dx}{x^2}$, it appears that $\frac{1}{x^2}$ is an integrating factor:

$$\frac{x\,dy - y\,dx}{x^2} = dx,$$
$$\frac{y}{x} = x + c.$$

(*c*) Solve the equation

$$x\,dy - y\,dx = x^3y\,dy.$$

Here we may write

$$\frac{x\,dy - y\,dx}{x^3} = y\,dy,$$

$$\frac{y}{x}\left(\frac{x\,dy - y\,dx}{x^2}\right) = y^2\,dy,$$

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 = \frac{1}{3}\,y^3 + c.$$

EXERCISES

Solve the following differential equations.

1. $(x + y + 1)\,dx + (x - y)\,dy = 0.$
2. $(x + 2\,y)\,dx + (2\,x + y)\,dy = 0.$
3. $4\,xy^2\,dx + 4\,x^2y\,dy + 2\,x\,dx + 3\,y^2\,dy = 0.$
4. $(\sin y + 2\,x)\,dx + x\cos y\,dy = 0.$
5. $(e^x - 2)\,dy + e^xy\,dx = 0.$ Solve by two methods.
6. $(y^2 - x^2)\,dx + 2\,xy\,dy = 0.$
7. $u(u + 2\,v)\,du + (u^2 - v^2)\,dv = 0.$
8. $x^2\,dy + 3\,xy\,dx = (x + 1)\,dx.$
9. $2\,x\,dy + y\,dx + dy = 0.$ Solve by two methods.
10. $2\,y^2\,dx + 3\,xy\,dy + x\,dy + y\,dx = 0.$
11. $x\,dy = (x + y)\,dx.$
12. $x\,dy - y\,dx = 2\,y^2\,dy.$
13. $x\,dy - y\,dx = y^3\,dx.$ Solve by two methods.
14. $x\,dy - y\,dx = (x^2 + y^2)\,dx.$
15. $x\,dy - y\,dx = \sqrt{x^2 - y^2}\,dx.$
16. $x\,dx + y\,dy = x\,dy - y\,dx.$
17. $x\,dy = (x^2y^2 - y)\,dx.$
18. $3\,xy^2\,dy + y^3\,dx + x\,dy = 0.$ Solve by two methods.
19. $2\,x\,dy - y\,dx = 2\,y^3\,dy.$
20. $x\,dy + y\,dx = (1 + x^2y^2)\,dy.$
21. $(x - y^2)\,dx + 2\,xy\,dy = 0.$
22. $(x^2 - y^2)\,dx + 2\,x\,dy = 2\,y\,dx.$
23. $3\,xy^2\,dy - y^3\,dx + 2\,x^2\,dx = 0.$
24. $(2\,y^2 - 3\,x)\,dx + 2\,xy\,dy = 0.$
25. $x\,dy - y\,dx = 2(y^2 + xy)\,dx.$
26. $4\,x\,dy - y\,dx = x^2\,dy.$
27. $4\,x\,dy + y\,dx = xy^2\,dy.$ Solve by two methods.
28. $x^2y\,dy + xy^2\,dx = (x^2y^2 - 1)dy.$
29. $(x^2 - y^2)\,dx + 2\,xy\,dy = 0.$
30. $x^2y^3\,dy + xy^2\,dy = x\,dy + y\,dx.$

184. The linear equation. A differential equation is said to be *linear* if it is *of the first degree in y and y'* (or, what amounts to the same thing, in y and dy). Every such equation may evidently be written in the form

$$dy + Py\,dx = Q\,dx, \tag{1}$$

where P and Q are functions of x alone.

Before undertaking to solve equation (1), let us consider the special case

$$dy + Py\,dx = 0. \tag{2}$$

Here the variables are separable, and the solution may be obtained at once:

$$\frac{dy}{y} + P\,dx = 0,$$

whence

$$\log y + \int P\,dx = c,$$

$$ye^{\int P\,dx} = c'. \tag{3}$$

Now, differentiating (3), we get

$$e^{\int P\,dx}(dy + Py\,dx) = 0,$$

which shows that $e^{\int P\,dx}$ is an integrating factor for equation (2). But since Q is a function of x alone, it follows that $e^{\int P\,dx}$ *is likewise an integrating factor for equation* (1).

Examples: (*a*) Solve the equation

$$dy + 2\,y\,dx = x\,dx.$$

Here

$$P = 2, \quad \int P\,dx = 2\,x, \quad e^{\int P\,dx} = e^{2x}.$$

Introducing the integrating factor e^{2x}, and integrating, we find

$$ye^{2x} = \int xe^{2x}\,dx = \tfrac{1}{2}\,xe^{2x} - \tfrac{1}{4}\,e^{2x} + c,$$

whence

$$y = \tfrac{1}{2}\,x - \tfrac{1}{4} + ce^{-2x}.$$

(*b*) Solve the equation

$$xy' - x^3 - y = 0.$$

Writing this in the form

$$dy - \frac{y}{x}\,dx = x^2\,dx, \tag{4}$$

we have

$$P = -\frac{1}{x}, \int P\,dx = -\log x,$$

whence

$$e^{\int P\,dx} = e^{-\log x} = \frac{1}{x},$$

by formula (5) of § 48. Hence, dividing equation (4) by x and integrating, we get

$$\frac{y}{x} = \int x\,dx = \frac{x^2}{2} + c,$$

$$2y = x^3 + c'x.$$

185. Equations linear in $f(y)$. The equation

$$f'(y)dy + Pf(y)dx = Q\,dx, \tag{1}$$

where P and Q are functions of x alone, is evidently *linear in* $f(y)$, and may be solved by the method of the preceding article.

An equation not given directly in the form (1) may sometimes be reduced to that form by a simple transformation. In particular, this is always possible with the equation

$$dy + Py\,dx = Qy^n\,dx.$$

The process is as shown in the following

Example: Solve the equation

$$y' + \frac{y}{x} = \frac{1}{y^2}.$$

Let us write the equation in the form

$$y^2\,dy + \frac{y^3}{x}\,dx = dx.$$

If we multiply through by 3, so that the first term becomes $d(y^3)$, this equation is seen to be linear in y^3:

$$3\,y^2\,dy + \frac{3\,y^3}{x}\,dx = 3\,dx.$$

Here

$$P = \frac{3}{x}, \quad \epsilon^{\int P\,dx} = e^{3\log x} = x^3,$$

whence the solution of the equation is

$$x^3y^3 = 3\int x^3\,dx = \tfrac{3}{4}\,x^4 + c.$$

EXERCISES

Solve the following equations.

1. $\dfrac{dy}{dx} + y = x.$ *Ans.* $y = x - 1 + ce^{-x}.$
2. $x\,dy - 3\,y\,dx = x^2\,dx.$ *Ans.* $y = cx^3 - x^2.$
3. $(x^3 + 4\,y)\,dx = x\,dy.$ *Ans.* $y = cx^4 - x^3.$
4. $y' - xy = x.$ Solve by two methods.
5. $(3\,y - x)\,dx + dy = 0.$ *Ans.* $y = \tfrac{1}{3}\,x - \tfrac{1}{9} + ce^{-3x}.$
6. $(x + 1)\,dy - 2\,y\,dx = (x + 1)^4\,dx.$
   Ans. $2\,y = (x + 1)^4 + c(x + 1)^2.$
7. $t\dfrac{dx}{dt} + x(1 + t) = e^t.$ *Ans.* $2\,xte^t = e^{2t} + c.$
8. $(x - 2\,y + 5)\,dx + (2\,x + 4)\,dy = 0.$
   Ans. $y = -\tfrac{1}{2}(x + 2)\log(x + 2) + \tfrac{3}{2} + c(x + 2).$
9. $y'\sin y + \sin x\cos y = \sin x.$ Solve in two ways.
10. $\dfrac{dv}{dt} = g - kv.$ Solve in two ways.
11. $(x + 3\,y - 1)\,dx = (1 - x)\,dy.$ *Ans.* $y = \dfrac{c}{(x - 1)^3} - \dfrac{1}{4}\,(x - 1).$
12. $\dfrac{dy}{dx} + y\cos x = \sin 2\,x.$ *Ans.* $y = 2\sin x - 2 + ce^{-\sin x}.$
13. $3\,y^2\,dy - 2\,y^3\,dx = (x + 1)\,dx.$ *Ans.* $y^3 = ce^{2x} - \tfrac{1}{2}\,x - \tfrac{3}{4}.$
14. $dy + y(1 - xy^2)\,dx = 0.$ *Ans.* $\dfrac{1}{y^2} = x + \dfrac{1}{2} + ce^{2x}.$
15. $xy\,dy - (x + 2\,y^2)\,dx = 0.$ *Ans.* $y^2 = -\tfrac{2}{3}\,x + cx^4.$
16. $x\,dy - y^3\,dx = 3\,y\,dx.$ Solve in two ways.
17. $(x + y^3)\,dx - y^2\,dy = 0.$ *Ans.* $3\,y^3 + 3\,x + 1 = ce^{3x}.$
18. $(xy^2 + y)\,dx - x\,dy = 0.$ *Ans.* $2\,x + x^2y = cy.$

19. $3\,y\,dx + (x + xy^2)\,dy = 0$. *Ans.* $x^3y = ce^{-\frac{1}{2}y^2}$.

20. $y\,dy + (xy^2 - x)\,dx = 0$. Solve in two ways.

21. $x\,dy + (xe^y - 1)\,dx = 0$. *Ans.* $2x = e^y(x^2 + c)$.

22. Solve Ex. 8, p. 308, by another method.

23. Find a particular solution of the equation $\frac{dv}{dt} = \cos t - v$ such that $v = 0$ when $t = 0$. *Ans.* $v = \frac{1}{2}(\sin t + \cos t) - \frac{1}{2}e^{-t}$.

24. Find a particular solution of the equation $v\frac{dv}{dx} = x - v^2$ satisfying the condition $v = 1$ when $x = 0$. *Ans.* $v^2 = x - \frac{1}{2} + \frac{3}{2}e^{-2x}$.

25. Find a particular solution of the equation $\frac{dv}{dx} = \frac{2v - x}{x}$ satisfying the condition $v = 0$ when $x = 1$. *Ans.* $v = x - x^2$.

MISCELLANEOUS EXERCISES

Solve the following equations.

1. $x^3\,dy = (1 + x^2y)\,dx$.

2. $dy - \sin x\,dx = 2\,y\,dx$.

3. $(x - y)\,dx = (x + 2y - 1)\,dy$.

4. $(1 + x^2)\,dy = (1 + xy)\,dx$.

5. $\frac{dv}{dt} = a - \cos kt$.

6. $\frac{dv}{dt} = 1 - v + \sin t$.

7. $\frac{dv}{dt} = a^2 - k^2v^2$.

8. $x\frac{dy}{dx} - y = x\sqrt{x^2 + y^2}$.

9. $x\,dy + (x^2 - y)\,dx = 0$.

10. $xy\,dy - 3\,y\,dx + x^2y\,dx = 2\,x\,dy$.

11. $x\,dy + 2\,y\,dx + x^4y^2\,dx = 0$.

12. $x\,dx - xy^2\,dy = dx + y^2\,dy$.

13. $x\,dx + xy^2\,dy = y^2\,dy$.

14. $2\,xy\,dx - x^2\,dy = y^2\,(dx + dy)$.

15. $x\,dx - x\,dy + 4\,y\,dx = 0$.

16. $x^2y\,dx + xy\,dy + 2\,y\,dx = x\,dy$.

17. $dy - dx + 2\,x\,dy = y^2\,dx$.

18. $2\,y^2\,dx - 2\,x^2\,dx + x^2\,dy = xy\,dx$.

Solve each of the following equations by two methods.

19. $vv' = 1 - v^2$.

20. $t\,dx + dt = t^2\,dx$.

21. $dy + x^2y\,dx = 0$.

22. $dy - x^2\,dx = 2\,x^2y\,dx$.

23. $(xy^3 - 2\,x)\,dx = (3\,x^2y^2 + y^2)\,dy$.

24. $dy - y\,dx + y\log y\,dx = 0$.

25. $y(1 - x^2y^3)\,dx + x\,dy = 0$.

26. $x^2\,dy - xy\,dx + a^2\,dy = 0$.

27. $(x^2 + y^2)\,dx + xy\,dy = 0$.

28. $(x^2 - y^2)\,dx + xy\,dy = 0$.

29. $x^2\,dy - y^2\,dx + xy\,dy = 0$.

30. $x^2\,dy + y^2\,dx + xy\,dy = 0$.

CHAPTER XXVI

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

186. General and particular solutions. Being given a relation between x, y, and n arbitrary constants, say

$$F(x, y, c_1, \cdots, c_n) = 0, \tag{1}$$

let us differentiate this equation n times in succession. The equations thus obtained form with the original relation a set of $n + 1$ equations from which the n constants may be eliminated. The result is a differential equation of the n-th order,

$$\Phi(x, y, y', \cdots, y^{(n)}) = 0. \tag{2}$$

Conversely, corresponding to a differential equation of the form (2), there exists in general a relation of the form (1) which satisfies the differential equation. Equation (1) is called the *general solution* of equation (2). Thus *the general solution of a differential equation of the n-th order involves n arbitrary constants* *. Equation (1) evidently represents, geometrically, a family of curves, the *integral curves* of (2).

A *particular solution* is one that is obtained from the general solution by assigning particular values to one or more of the arbitrary constants. Thus a particular solution may contain any number of constants less than the maximum number, n.

* It is understood that the general solution contains n *essential* constants: *i.e.* that it cannot be replaced by an equally general form containing a smaller number of constants. Thus the equation

$$y = c_1 e^{x+c_2}$$

appears at first sight to contain two constants, but there is really only one. For, writing the equation in the form

$$y = c_1 e^x \cdot e^{c_2}$$

and setting $c_1 e^{c_2} = C$, we see that the equation

$$y = Ce^x$$

is equally general.

For example, it follows from Ex. 47, p. 54, that the equation

$$\frac{d^2x}{dt^2} = -k^2x$$

is satisfied by the equation

$$x = A\cos kt + B\sin kt,$$

where A and B are arbitrary. Since the differential equation is of the second order, the solution here given, containing two constants, is the general solution. Particular solutions are

$$x = A\cos kt,$$
$$x = A(\cos kt + \sin kt),$$
$$x = 2\cos kt - 3\sin kt,$$
$$x = 0,$$

etc.

EXERCISES

Find the differential equation whose general solution is as follows.

1. $y = c_1 + c_2e^{2x}$. *Ans.* $y'' - 2y' = 0$.
2. $y = c_1e^x + c_2e^{-x}$. *Ans.* $y'' = y$.
3. $y = c_1 + c_2x$.
4. $y = c_1e^x + c_2e^{5x}$.
5. $y = c_1\sin x + c_2\cos x$.
6. $y = c_1\sin 2x + c_2\cos 2x$.
7. $x = c_1 + c_2t + t^2$.
8. $x = c_1\log t + c_2$.
9. $x = t + c_1\log t + c_2$. *Ans.* $t\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 1$.
10. $y = c_1\cos x + c_2\sin x - \frac{1}{3}\cos 2x$. *Ans.* $y'' + y = \cos 2x$.
11. $y = c_1 + c_2x + c_3x^2$. *Ans.* $y''' = 0$.
12. $y = c_1 + c_2e^x + c_3e^{-x}$. *Ans.* $y''' = y'$.
13. $y = c_1 + c_2e^{2x} + c_3xe^{2x}$. *Ans.* $y''' - 4y'' + 4y' = 0$.
14. $y = c_1e^x + c_2e^{-x} + c_3 + c_4x$. *Ans.* $y^{(4)} - y'' = 0$.
15. $y = c_1 + c_2e^{-x}\cos x + c_3e^{-x}\sin x$. *Ans.* $y''' + 2y'' + 2y' = 0$.

Find the differential equation of each of the following families of curves.

16. All straight lines. *Ans.* $y'' = 0$.
17. Parabolas with axis lying in Ox. *Ans.* $yy'' + y'^2 = 0$.
18. Parabolas with axis parallel to Oy. *Ans.* $y''' = 0$.
19. Circles of radius a. (Cf. Ex. 19, p. 103.) *Ans.* $ay'' = (1 + y'^2)^{\frac{3}{2}}$.
20. All circles. (Cf. Ex. 19.) *Ans.* $(1 + y'^2)y''' = 3y'y''^2$.

187. The equation $y'' = f(x)$. In the remainder of this chapter we will develop methods for solving various types of differential equations, chiefly of the second order.

The simplest equation of the second order is that in which the second derivative is a function of the independent variable — say

$$y'' = f(x).$$

This equation can evidently be solved by two successive integrations. In fact, it is obvious that the equation

$$y^{(n)} = f(x)$$

can be solved by n successive integrations.

188. Dependent variable absent. Given an equation of the second order in which the *dependent variable* y does not occur — *i.e.* an equation involving x, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ — let us put

$$\frac{dy}{dx} = v, \quad \frac{d^2y}{dx^2} = \frac{dv}{dx}.$$

We thus obtain an equation of the *first order* in x and v. Solving this by Chapter XXV, replacing v by $\frac{dy}{dx}$, and solving again by Chapter XXV, we obtain the general solution of the original equation.

The problem of § 187 is evidently a special case of the present one.

Example: Solve the equation

$$(1 + x)y'' - y' = 0.$$

Setting

$$y' = v, \quad y'' = \frac{dv}{dx},$$

we have

$$(1 + x)dv - v\,dx = 0,$$

or

$$\frac{dv}{v} - \frac{dx}{1 + x} = 0.$$

Hence

$$\log v - \log (1 + x) = c_1,$$

or

$$v = c_1'(1 + x).$$

Replacing v by y', and integrating again, we find

$$y = \tfrac{1}{2} c_1'(1 + x)^2 + c_2,$$

or, with c_1 in place of $\frac{1}{2} c_1'$,

$$y = c_1(1 + x)^2 + c_2.$$

189. Independent variable absent. An equation of the second order from which the *independent variable* x is absent may be written as an equation of the first order in the variables y and v by putting

$$\frac{dy}{dx} = v, \quad \frac{d^2y}{dx^2} = v\frac{dv}{dy},$$

after which the process of solution is analogous to that of § 188. The formula just written for the second derivative is established as follows:

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v\frac{dv}{dy}.$$

Example: Solve the equation

$$y'' = yy'.$$

With

$$y' = v, \; y'' = v\frac{dv}{dy},$$

this becomes

$$v\frac{dv}{dy} = yv,$$

or

$$dv = y\,dy,$$
$$v = \tfrac{1}{2} y^2 + c_1.$$

Whence

$$\frac{2\,dy}{y^2 + 2c_1} = dx,$$

and, if $c_1 > 0$,

$$\sqrt{\frac{2}{c_1}} \arctan \frac{y}{\sqrt{2c_1}} = x + c_2.$$

EXERCISES

Solve the following equations.

1. $y'' = x^3$.
2. $y'' = \cos 3x$.
3. $t\dfrac{d^2x}{dt^2} = 1$.
4. $\dfrac{d^2s}{dt^2} = -\dfrac{1}{t^2}$.
5. $xy'' + y' = 0$.
6. $xy'' - y' = 0$.
7. $t\dfrac{d^2x}{dt^2} - (t^2+1)\dfrac{dx}{dt} = 0$. *Ans.* $x = c_1 e^{\frac{1}{2}t^2} + c_2$.
8. $t\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 1$. *Ans.* $x = t + c_1 \log t + c_2$.
9. $xy'' + y' = x$. *Ans.* $y = \frac{1}{4}x^2 + c_1 \log x + c_2$.
10. $(1-x^2)y'' - xy' = 2x$. *Ans.* $y = c_1 \arcsin x - 2x + c_2$.
11. $xy'' - y' = x$. *Ans.* $y = \frac{1}{2}x^2 \log x + c_1 x^2 + c_2$.
12. $\dfrac{dx}{dt}\cdot\dfrac{d^2x}{dt^2} = t$.
Ans. $x = \frac{1}{3}t\sqrt{t^2+c_1} + \frac{1}{2}c_1 \log\left(t + \sqrt{t^2+c_1}\right) + c_2$.
13. $t^2\dfrac{d^2x}{dt^2} + 2t\dfrac{dx}{dt} = 1$. *Ans.* $x = \log t + \dfrac{c_1}{t} + c_2$.
14. $t\dfrac{d^2x}{dt^2} = 2\dfrac{dx}{dt} + t^3$. *Ans.* $x = \frac{1}{4}t^4 + c_1 t^3 + c_2$.
15. $yy'' = y'^2$. *Ans.* $x = c_1 \log y + c_2$.
16. $\dfrac{d^2x}{dt^2} = 2x\dfrac{dx}{dt}$. *Ans.* $\dfrac{1}{c_1}\arctan\dfrac{x}{c_1} = t + c_2$.
17. $x\dfrac{d^2x}{dt^2} + 2\left(\dfrac{dx}{dt}\right)^2 = 0$. *Ans.* $x^3 = c_1 t + c_2$.
18. $x\dfrac{d^2x}{dt^2} - 2\left(\dfrac{dx}{dt}\right)^2 = 0$. *Ans.* $\dfrac{1}{x} = c_1 t + c_2$.
19. $y^3y'' = 1$. *Ans.* $\sqrt{c_1y^2 - 1} = c_1x + c_2$.
20. $y^2y'' = y'$. *Ans.* $x = c_1^2 \log(c_1 + y) - c_1 y + c_2$.
21. $y'' + yy' = 0$. *Ans.* $\dfrac{c_1 + y}{c_1 - y} = c_2 e^{c_1 x}$.
22. $yy'' + y'^2 = 1$. *Ans.* $c_1^2 + y^2 = (x + c_2)^2$.
23. $y'' - y'^2 = 1$. *Ans.* $y = -\log\cos(x + c_1) + c_2$.
24. $\dfrac{d^2y}{dt^2} = \sqrt{1 - \left(\dfrac{dy}{dt}\right)^2}$. *Ans.* $y = -\cos(t + c_1) + c_2$.
25. $y'' + y'^2 + 1 = 0$. *Ans.* $y = \log\cos(x + c_1) + c_2$.
26. $y'' = y' + y'^3$. *Ans.* $x = \log\sin(y + c_1) + c_2$.
27. $\dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^2 = 0$. (Two methods.) *Ans.* $x = \log(t + c_1) + c_2$.
28. $\dfrac{dy}{dt}\cdot\dfrac{d^2y}{dt^2} = 1$. (Two methods.) *Ans.* $9(y + c_2)^2 = 8(t + c_1)^3$.

190. The linear equation. A differential equation of the n-th order is said to be *linear* if it is of *the first degree in* $y, y', \cdots, y^{(n)}$ (cf. § 184). Thus every linear differential equation of the n-th order can be written in the form

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = X,$$

where the coefficients $p_1, \cdots, p_n$ and the right member X are functions of x.

In what follows, we shall be concerned entirely with the special case in which the functions $p_1, \cdots, p_n$ are *constants:*

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = X. \tag{1}$$

This equation is of frequent occurrence in the applications.

191. The homogeneous linear equation. A linear differential equation whose right-hand member is 0 is said to be *homogeneous**. Thus the general form of the homogeneous linear equation with constant coefficients is

$$\boldsymbol{y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.} \tag{1}$$

If $y = y_1$ is a particular solution of equation (1), then $y = c_1 y_1$, where c_1 is arbitrary, is also a solution, as appears at once by substitution in (1). Further, if $y = y_2$ is a second particular solution, then not only $y = c_2 y_2$ but also

$$y = c_1 y_1 + c_2 y_2$$

is a solution. Finally, if

$$y = y_1, \; y = y_2, \; \cdots, \; y = y_n$$

are n distinct particular solutions, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is a solution, and since it contains n arbitrary constants, *it is the general solution.*

We proceed to show that the general solution of equation (1) can always be written down, provided a certain algebraic equation of the n-th degree can be solved. The theory will be developed in detail only for the equation of the second order.

* That is, it is homogeneous in y and its derivatives. See § 181.

192. The characteristic equation. The homogeneous linear equation of the first order, viz.,

$$y' + a_1 y = 0,$$

is evidently satisfied by

$$y = e^{-a_1 x}.$$

This suggests the possibility of *determining* m so that

$$y = e^{mx}$$

will be a solution of the equation

$$y'' + a_1 y' + a_2 y = 0. \tag{1}$$

Substituting in (1) the values

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

and bracketing out the factor e^{mx}, we find that the differential equation is satisfied, provided

$$m^2 + a_1 m + a_2 = 0. \tag{2}$$

Equation (2) is called the *characteristic equation* corresponding to (1). Thus

$$y = e^{mx}$$

is a solution of equation (1) *if and only if* m *is a root of the characteristic equation.*

193. Distinct roots. If the roots m_1, m_2 of the characteristic equation are distinct, we obtain at once two distinct particular solutions of the differential equation, viz.,

$$y = e^{m_1 x}, \quad y = e^{m_2 x}.$$

Hence, by § 191, the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{1}$$

Example: Solve the differential equation

$$2y'' - y' - y = 0.$$

The characteristic equation is

$$2m^2 - m - 1 = 0,$$

whence

$$m = 1 \text{ or } -\tfrac{1}{2}.$$

Thus the general solution of the given equation is

$$y = c_1 e^{x} + c_2 e^{-\frac{1}{2}x}.$$

EXERCISES

Solve the following differential equations.

1. $y'' - 5y' + 6y = 0$. *Ans.* $y = c_1e^{2x} + c_2e^{3x}$.

2. $2\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + y = 0$. *Ans.* $y = c_1e^{\frac{1}{2}t} + c_2e^{t}$.

3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

4. $y'' + 12y' + 32y = 0$.

5. $4\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + x = 0$.

6. $3\frac{d^2r}{dz^2} + 7\frac{dr}{dz} + 2r = 0$.

7. $y'' + 3y' = 0$.

8. $y'' + 11y' + 28y = 0$.

9. $6\frac{d^2y}{dx^2} = \frac{dy}{dx} + y$.

10. $\frac{d^2x}{dt^2} + k\frac{dx}{dt} = 0$.

11. $\frac{d^2\theta}{dt^2} = \theta$.

12. $\frac{d^2x}{dt^2} = k^2x$.

13. $y'' - 4y' + 2y = 0$.

14. $5y'' - 7y' - y = 0$.

15. $20\frac{d^2z}{dx^2} - 9\frac{dz}{dx} = 18z$.

16. $z = 4\frac{dz}{dt} + 5\frac{d^2z}{dt^2}$.

For the following differential equations, find a particular solution satisfying the given conditions.

17. $y'' + 2y' = 0$; $y = 0$ when $x = 0$. *Ans.* $y = c_1(1 - e^{-2x})$.

18. $y'' + 2y' = 0$; $y' = 1$ when $x = 0$. *Ans.* $y = c_1 - \frac{1}{2}e^{-2x}$.

19. $y'' + 2y' = 0$; $y = 0$, $y' = 1$ when $x = 0$. *Ans.* $y = \frac{1}{2} - \frac{1}{2}e^{-2x}$.

20. $y'' + 2y' = 0$; $y = 0$ when $x = 0$, $y = 1$ when $x = -1$. *Ans.* $(1 - e^2)y = 1 - e^{-2x}$.

21. $y'' + y' = 2y$; $y = 0$, $y' = 0$ when $x = 0$. *Ans.* $y = 0$.

22. $y'' = 4y$; $y = 1$, $y' = -1$ when $x = 0$.

23. $y'' + 5y' + 6y = 0$; $y = 3$, $y' = 0$ when $x = 0$.

24. $y'' + y' = 2y$; $y = 0$ when $x = 0$, $y = 2$ when $x = 1$.

25. $y'' = y$; $y' = 0$ when $x = 0$, $y' = 3$ when $x = 1$.

26. Determine the curves for which the rate of change of the slope is equal to the slope.

27. In Ex. 26, find the curve that touches the line $y = 2x$ at the origin. *Ans.* $y = 2e^x - 2$.

28. In Ex. 26, find the curves that cross the y-axis at 45°. *Ans.* $y = e^x + c_1$.

29. In Ex. 26, find the curve that passes through (0, 1) and approaches the negative x-axis asymptotically. *Ans.* $y = e^x$.

30. Find the equation of a curve for which $y'' = y$, if it crosses the y-axis at right angles at (0, 1). *Ans.* $y = \cosh x$.

194. Equal roots. When the characteristic equation has equal roots, the method of the previous article does not give the general solution. For, if $m_1 = m_2$, equation (1) above becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x}$$
$$= (c_1 + c_2) e^{m_1 x} = c' e^{m_1 x};$$

hence the solution contains only a single constant, and is a particular solution.

To find a second particular solution, let us try

$$y = x e^{m_1 x},$$

whence

$$y' = e^{m_1 x}(m_1 x + 1),$$
$$y'' = e^{m_1 x}(m_1^2 x + 2\, m_1).$$

Substituting in the differential equation, we find that

$$y = x e^{m_1 x}$$

will be a solution, provided

$$(1) \qquad (m_1^2 + a_1 m_1 + a_2)x + 2\, m_1 + a_1 \equiv 0.$$

Now the coefficient of x vanishes because m_1 is a root of the characteristic equation. Further, since $m_1 = m_2$, it follows that

$$m_1 = -\tfrac{1}{2} a_1,$$

or

$$\mathbf{2\, m_1 + a_1 = 0.}$$

Thus (1) holds, and $y = x e^{m_1 x}$ is a second particular solution.

Therefore the general solution of the differential equation in the case of equal roots is

$$\boldsymbol{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}.}$$

195. The imaginary exponential function. In § 150 we established the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

valid for all (real) values of x. Replacing x by ix, we obtain, *formally*,

$$(1) \qquad e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots.$$

In § 150 and Ex. 1, p. 249, we also found

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \tag{2}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots. \tag{3}$$

Upon multiplying both members of (3) by i, it appears that (1), (2), and (3) are connected by the relation

$$e^{ix} = \cos x + i \sin x. \tag{4}$$

In the theory of functions of a complex variable, this formula is taken as the *definition* of the function e^{ix}.

196. Complex roots. In the problem of § 193, if the characteristic equation has *complex roots* $\alpha \pm i\beta$, the general solution takes the form

$$\begin{aligned} y &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}). \end{aligned} \tag{1}$$

By means of (4), § 195, the right member of (1) may be simplified. For,

$$\begin{aligned} e^{i\beta x} &= \cos \beta x + i \sin \beta x, \\ e^{-i\beta x} &= \cos \beta x - i \sin \beta x. \end{aligned}$$

Whence (1) becomes

$$y = e^{\alpha x}[(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x],$$

or, if we place

$$c_1 + c_2 = c_1', \quad i(c_1 - c_2) = c_2'$$

and drop the accents,

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \tag{3}$$

Changing the constants again, we may write (3) in the form

$$y = Ae^{\alpha x} \cos (\beta x + \epsilon), \tag{4}$$

where the arbitrary constants A and ϵ are given in terms of c_1 and c_2 by the formulas

$$A = \pm\sqrt{c_1^2 + c_2^2}, \tag{5}$$

$$\epsilon = -\arctan \frac{c_2}{c_1}, \tag{6}$$

as is easily verified. The form (4) is to be preferred in certain applications.

Example: Find a particular solution of the equation

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$$

satisfying the conditions $x = 1$, $\frac{dx}{dt} = 5$ when $t = 0$.

The characteristic equation is

$$m^2 - 4m + 13 = 0,$$

whence

$$m = 2 \pm \sqrt{-9} = 2 \pm 3i.$$

Thus the general solution is, in the form (3),

(7) $$x = e^{2t}(c_1 \cos 3t + c_2 \sin 3t).$$

To find the desired particular solution, differentiate (7):

(8) $$\frac{dx}{dt} = e^{2t}(2c_1 \cos 3t + 2c_2 \sin 3t - 3c_1 \sin 3t + 3c_2 \cos 3t).$$

Substituting $x = 1$, $t = 0$ in (7) and $\frac{dx}{dt} = 5$, $t = 0$ in (8), we get

$$1 = c_1, \quad 5 = 2c_1 + 3c_2,$$

whence $c_1 = 1$, $c_2 = 1$, and the solution is

$$x = e^{2t}(\cos 3t + \sin 3t).$$

In the form (4) this becomes, as appears from* (5) and (6),

$$x = \sqrt{2}\, e^{2t} \cos (3t - \tfrac{1}{4}\pi).$$

EXERCISES

Solve the following equations.

1. $y'' - 4y' + 4y = 0$. *Ans.* $y = c_1e^{2x} + c_2xe^{2x}$.
2. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$. *Ans.* $x = c_1e^{-3t} + c_2te^{-3t}$.
3. $y'' + 2y' + y = 0$.
4. $9y'' + 12y' + 4y = 0$.
5. $\frac{d^2\theta}{dt^2} = 0$.
6. $4\frac{d^2r}{dt^2} + 4\frac{dr}{dt} + r = 0$.
7. $y'' - 2ky' + k^2y = 0$.
8. $4y'' + 25y = 20y'$.
9. $y'' + 2y' + 5y = 0$. *Ans.* $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$.
10. $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 10x = 0$. *Ans.* $x = c_1e^{3t} \cos (t + c_2)$.

* The sign to be chosen in (5) must be determined in accordance with the initial conditions: in this case, so as to make x positive when $t = 0$.

11. $\frac{d^2y}{dx^2} + 9y = 0.$ **12.** $\frac{d^2x}{dt^2} = -k^2x.$

13. $8y'' + 16y' + 9y = 0.$ **14.** $2y'' - 5y' = 5y.$

15. $6y'' + y = 0.$ **16.** $y'' - 5y' = 5y.$

For the following differential equations, find a particular solution satisfying the given conditions.

17. $y'' - 2y' + y = 0$; $y = 0$ when $x = 0$. *Ans.* $y = cxe^x$.

18. $y'' - 2y' + y = 0$; $y = 1$, $y' = 0$ when $x = 0$.

19. $y'' - 2y' + y = 0$; $y = 0$ when $x = 0$, $y = 2$ when $x = 1$.

Ans. $y = 2xe^{x-1}$.

20. $y'' - 2y' + y = 0$; $y = 2$ when $x = 1$, $y = -3$ when $x = 2$.

Ans. $y = \left(\frac{4}{e} + \frac{3}{e^2}\right)e^x - \left(\frac{2}{e} + \frac{3}{e^2}\right)xe^x$.

In Exs. 21–30, write the answer in both forms of § 196.

21. $\frac{d^2x}{dt^2} = -4x$; $x = 2$, $\frac{dx}{dt} = 0$ when $t = 0$. *Ans.* $x = 2\cos 2t$.

22. $\frac{d^2x}{dt^2} = -4x$; $x = 1$, $\frac{dx}{dt} = 2$ when $t = 0$.

Ans. $x = \sin 2t + \cos 2t = \sqrt{2}\cos(2t - \frac{1}{4}\pi)$.

23. $\frac{d^2x}{dt^2} = -4x$; $x = 2$, $\frac{dx}{dt} = -6$ when $t = 0$.

Ans. $x = 2\cos 2t - 3\sin 2t = 3.61\cos(2t + 0.98)$.

24. $\frac{d^2x}{dt^2} = -k^2x$; $x = a$, $\frac{dx}{dt} = b$ when $t = 0$.

Ans. $x = a\cos kt + \frac{b}{k}\sin kt = \sqrt{a^2 + \frac{b^2}{k^2}}\cos(kt + \epsilon)$, $\epsilon = -\arctan\frac{b}{ak}$.

25. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$; $x = -1$, $\frac{dx}{dt} = 0$ when $t = 0$.

Ans. $x = -e^{-t}(\cos t + \sin t) = -\sqrt{2}\,e^{-t}\cos(t - \frac{1}{4}\pi)$.

26. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$; $x = -1$, $\frac{dx}{dt} = 2$ when $t = 0$.

Ans. $x = e^{-t}(-\cos t + \sin t) = -\sqrt{2}\,e^{-t}\cos(t + \frac{1}{4}\pi)$.

27. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0$; $x = -1$, $\frac{dx}{dt} = 8$ when $t = 0$.

Ans. $x = e^{-2t}(-\cos 2t + 3\sin 2t) = -\sqrt{10}\,e^{-2t}\cos(2t + 1.25)$.

28. $\frac{d^2x}{dt^2} = -5x - 4\frac{dx}{dt}$; $x = a$, $\frac{dx}{dt} = 0$ when $t = 0$.

Ans. $x = ae^{-2t}(\cos t + 2\sin t) = \sqrt{5}\,ae^{-2t}\cos(2t - 1.11)$.

29. $\frac{d^2x}{dt^2} = -\frac{1}{4}x$; $x = 4$ when $t = 0$, $x = 1$ when $t = \pi$.

Ans. $x = 4\cos\frac{1}{2}t + \sin\frac{1}{2}t = \sqrt{17}\cos(\frac{1}{2}t - 0.24)$.

30. $\dfrac{d^2x}{dt^2} = -4.36\,x$; $x = -1$ when $t = 0$, $x = 2.35$ when $t = 1$.

Ans. $x = -\cos 2.09\,t + 2.14 \sin 2.09\,t = -2.36 \cos (2.09\,t + 1.13)$.

31. Find the equation of a curve for which $y'' = -y$, if it touches the line $y = x + 1$ at $(0, 1)$. *Ans.* $y = \sin x + \cos x$.

32. Using formula (4) of § 195, show that

$$e^{\frac{1}{2}\pi i} = i, \quad e^{\pi i} = -1, \quad e^{2\pi i} = 1.$$

33. Show that $\log(-1) = \pi i$, $\log(-k) = \log k + \pi i$.

34. Show that, if n is an integer, $e^{2n\pi i} = 1$, whence $\log 1 = 2n\pi i$.

35. We may write

$$\int \frac{dx}{x - a} = \log (x - a) + c,$$

or

$$\int \frac{dx}{x - a} = \int \frac{-dx}{a - x} = \log (a - x) + c.$$

Show that the two results agree.

197. Extension to equations of higher order. The theory of §§ 192–196 is readily extended to equations of order higher than the second. We give the results without proof:

Let there be given a differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0. \tag{1}$$

(*a*) If the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

are all distinct, the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

(*b*) Corresponding to a *double root* m_1, the terms in the general solution are

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x};$$

corresponding to a triple root,

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x}; \text{ etc.}$$

(*c*) A pair of complex roots $\alpha \pm i\beta$ give rise to the terms

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x);$$

a pair of *double roots* $\alpha \pm i\beta$ give rise to the terms

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x + c_3 x \cos \beta x + c_4 x \sin \beta x); \text{ etc.}$$

EXERCISES

Solve the following equations.

1. $y''' = 4\,y'$. *Ans.* $y = c_1 + c_2e^{2x} + c_3e^{-2x}$.
2. $y''' + 4\,y'' + 4\,y' = 0$. *Ans.* $y = c_1 + c_2e^{-2x} + c_3xe^{-2x}$.
3. $y''' = y'' + 6\,y'$.
4. $y''' - 7\,y' + 6\,y = 0$.
5. $\dfrac{d^3y}{dx^3} = \dfrac{d^2y}{dx^2}$.
6. $\dfrac{d^3x}{dt^3} + 4\dfrac{dx}{dt} = 0$.
7. $y''' - 6\,y'' + 13\,y' = 0$.
8. $4\,y''' - 3\,y' + y = 0$.
9. $\dfrac{d^3x}{dt^3} = x$. *Ans.* $x = c_1e^t + e^{-\frac{1}{2}t}\,(c_2 \cos \tfrac{1}{2}\sqrt{3}\,t + c_3 \sin \tfrac{1}{2}\sqrt{3}\,t)$.
10. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} = y$.
Ans. $y = c_1e^x + c_2e^{-x} + c_3xe^{-x} + c_4x^2e^{-x}$.
11. $y^{(4)} - 12\,y'' + 27\,y = 0$.
12. $y^{(4)} - 2\,y'' + y = 0$.
13. $y''' + 3\,y'' + 3\,y' + y = 0$.
14. $y''' - 2\,y'' - y' + 2\,y = 0$.
15. $y^{(4)} + 2\,y'' + y = 0$.
16. $y^{(4)} - 2\,y''' + 2\,y'' - 2\,y' + y = 0$.
17. $6\dfrac{d^5x}{dt^5} + 35\dfrac{d^4x}{dt^4} + 47\dfrac{d^3x}{dt^3} + 12\dfrac{d^2x}{dt^2} = 0$.
Ans. $x = c_1 + c_2t + c_3e^{-4t} + c_4e^{-\frac{1}{3}t} + c_5e^{-\frac{3}{2}t}$.
18. $y^{(4)} - 4\,y''' + 14\,y'' - 20\,y' + 25\,y = 0$.
Ans. $y = e^x(c_1 \cos 2\,x + c_2 \sin 2\,x + c_3x \cos 2\,x + c_4x \sin 2\,x)$.
19. $y^{(4)} - 7\,y''' + 8\,y'' + 28\,y' - 48\,y = 0$.
20. Find a particular solution of the equation $\dfrac{d^3x}{dt^3} = 0$ satisfying the conditions $x = 0$ when $t = 0$, $x = 2$ when $t = 1$, $\dfrac{dx}{dt} = 1$ when $t = 0$.
Ans. $x = t + t^2$.
21. Find a particular solution of the equation $y''' + 3\,y'' = 0$ satisfying the conditions $y = 0$, $y' = 0$, and $y'' = 1$ when $x = 0$.
Ans. $y = \frac{1}{9}\,e^{-3x} + \frac{1}{3}\,x - \frac{1}{9}$.
22. Find a particular solution of the equation $y''' + 4\,y' = 0$ such that $y = 0$ when $x = 0$, $y = -1$ when $x = \frac{1}{4}\,\pi$, $y = 2$ when $x = \frac{1}{2}\,\pi$.
Ans. $y = 1 - \cos 2\,x - 2 \sin 2\,x$.
23. Find a particular solution of the equation $\dfrac{d^3x}{dt^3} + 3\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 3\,x = 0$ satisfying the conditions $x = 0$, $\dfrac{dx}{dt} = 4$, $\dfrac{d^2x}{dt^2} = 0$ when $t = 0$.
Ans. $x = 2\,e^t - 2\,e^{-t}$.

198. The non-homogeneous linear equation. Let us consider now the *non-homogeneous* linear equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = X. \tag{1}$$

In solving this equation, the first step is to write down the general solution

$$Y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the *homogeneous* equation obtained from (1) by *making the right member* 0. The quantity Y is called the *complementary function.*

The next step is to obtain, by any means whatever, a *particular integral* of (1), say

$$y = \overline{y}.$$

Then the equation

$$y = Y + \overline{y} \tag{2}$$

is a solution of (1), as appears by substitution, and since it contains n arbitrary constants, it is the general solution.

Various methods exist for finding the particular solution

$$y = \overline{y}.$$

The method given below, though not entirely general, is usually the best method when it applies, and it is available in nearly all of the simpler applications.

We begin with an

Example: Solve the equation

$$y'' - 5\,y' + 6\,y = x + e^{2x}. \tag{3}$$

The complementary function, *i.e.* the solution of the equation

$$y'' - 5\,y' + 6\,y = 0,$$

is

$$Y = c_1 e^{2x} + c_2 e^{3x}.$$

To obtain a particular integral of (3), proceed as follows: Differentiating twice, we obtain

$$y^{(4)} - 5\,y''' + 6\,y'' = 4\,e^{2x}. \tag{4}$$

Differentiating again, we get

$$y^{(5)} - 5\,y^{(4)} + 6\,y''' = 8\,e^{2x}. \tag{5}$$

Multiplying equation (4) by 2 and subtracting from (5), we get the *homogeneous* equation

$$(6) \quad y^{(5)} - 7\,y^{(4)} + 16\,y''' - 12\,y'' = 0.$$

It is easily seen that the complementary function Y forms part of the solution of this equation; hence two of the roots of the characteristic equation

$$m^5 - 7\,m^4 + 16\,m^3 - 12\,m^2 = 0$$

are $m = 2, 3$. The other roots are 2, 0, 0. Thus the general solution of (6) is

$$(7) \qquad y = c_1e^{2x} + c_2e^{3x} + c_3 + c_4x + c_5xe^{2x}.$$

Let us substitute y in the original equation as a trial solution, noting, however, that the terms arising from the complementary function must disappear identically after the substitution, so that it is sufficient to substitute*

$$(8) \qquad y = c_3 + c_4x + c_5xe^{2x}.$$

We have

$$y' = c_4 + 2\,c_5xe^{2x} + c_5e^{2x},$$
$$y'' = 4\,c_5xe^{2x} + 4\,c_5e^{2x}.$$

Substituting in (3), we find that (8) will be a particular integral provided the equation

$$4\,c_5xe^{2x} + 4\,c_5e^{2x} - 5\,c_4 - 10\,c_5xe^{2x} - 5\,c_5e^{2x} + 6\,c_3 + 6\,c_4x + 6\,c_5xe^{2x} = x + e^{2x}$$

holds identically — *i.e.* for all values of x. The terms in xe^{2x} destroy each other. Equating coefficients of the other functions, we find the following:

Coefficients of e^{2x}: $\quad 4\,c_5 - 5\,c_5 = 1.$
Coefficients of x: $\quad 6\,c_4 = 1.$
Constant terms: $\quad -5\,c_4 + 6\,c_3 = 0.$

This gives

$$c_5 = -1, \quad c_4 = \tfrac{1}{6}, \quad c_3 = \tfrac{5}{36}.$$

Substituting in (7), we get as the general solution of (3)

$$y = c_1e^{2x} + c_2e^{3x} + \tfrac{5}{36} + \tfrac{1}{6}\,x - xe^{2x}.$$

* That is, we place, temporarily, $c_1 = c_2 = 0$. This is of course allowable since we are trying merely to find the particular solution y

Thus the method consists of the following steps:

(*a*) Write down the complementary function.

(*b*) Differentiate both members of the given equation successively until the right member becomes 0, either directly or by elimination. The original equation is thus replaced by a derived *homogeneous* equation of higher order (equation (6) in the example).

(*c*) Write down, by § 197, the general solution (equation (7) above) of this derived equation. The complementary function will always be a part of this solution, so that certain of the roots of the characteristic equation are known beforehand; these should be removed at once by division.

(*d*) Of the arbitrary constants occurring in this general solution, those belonging to the complementary function (c_1, c_2 above) will remain arbitrary in the final result; they may therefore be placed temporarily equal to 0, since we are trying to find merely a particular solution of the original equation. The other constants, the so-called *superfluous constants*, are determined by substituting the value of y in the original equation as a trial solution and equating coefficients.

It is clear that the success of the method depends on our ability to reduce the right-hand member to 0 by differentiation and elimination, as in the above example. Hence X and its successive derivatives must contain only a finite number of distinct functions of x. The method therefore applies whenever X contains only constants or terms of the form x^n, e^{ax}, $\sin^n ax$, $\cos^n ax$, or products of these, n being a positive integer.

EXERCISES

Solve the following equations.

1. $y'' - 3y' + 2y = 3x - 2$. *Ans.* $y = c_1e^x + c_2e^{2x} + \frac{3}{2}x + \frac{5}{4}$.
2. $2y'' - y' = x - 5$. *Ans.* $y = c_1e^{\frac{1}{2}x} + c_2 + 3x - \frac{1}{2}x^2$.
3. $\frac{d^2z}{dt^2} - 5\frac{dz}{dt} + 6z = e^{4t}$. *Ans.* $z = c_1e^{3t} + c_2e^{2t} + \frac{1}{2}e^{4t}$.
4. $y'' + y' = (1 + x)^2$. *Ans.* $y = c_1e^{-x} + c_2 + x + \frac{1}{3}x^3$.

5. $\frac{d^2x}{dt^2} + x = \cos 2t.$ *Ans.* $x = c_1 \sin t + c_2 \cos t - \frac{1}{3} \cos 2t.$

6. $3y'' + 2y = 2e^{-x} - 2.$ *Ans.* $y = \text{C. F.} + \frac{2}{5} e^{-x} - 1.$

7. $\frac{d^2x}{dt^2} = x + 5.$ *Ans.* $x = c_1 e^t + c_2 e^{-t} - 5.$

8. $\frac{d^2u}{dv^2} - v = 1.$ *Ans.* $u = \frac{1}{6} v^3 + \frac{1}{2} v^2 + c_1 v + c_2.$

9. $4y'' - 5y' + y = 2 \sin 3x - 5 \cos 3x.$

Ans. $y = \text{C. F.} + \frac{41}{290} \cos 3x + \frac{1}{290} \sin 3x.$

10. $y'' + y = e^{2x} + 2e^x.$ *Ans.* $y = \text{C. F.} + \frac{1}{5} e^{2x} + e^x.$

11. $\frac{d^2x}{dt^2} = \cos t - x.$ *Ans.* $x = \text{C. F.} + \frac{1}{2} t \sin t.$

12. $y'' - 2y' + y = 2e^{3x} - 3x + 4.$

Ans. $y = \text{C. F.} + \frac{1}{2} e^{3x} - 3x - 2.$

13. $y'' - 5y' + 6y = \cos x - e^{2x}.$

Ans. $y = \text{C. F.} + \frac{1}{10} \cos x - \frac{1}{10} \sin x + xe^{2x}.$

14. $\frac{d^2x}{dt^2} + 4x = \sin 3t + t^2.$ *Ans.* $x = \text{C. F.} - \frac{1}{5} \sin 3t + \frac{1}{4} t^2 - \frac{1}{8}.$

15. $y'' - y = e^x \sin x.$ *Ans.* $y = \text{C. F.} - \frac{2}{5} e^x \cos x - \frac{1}{5} e^x \sin x.$

16. $y'' + y = 4 \sin x \cos x.$ *Ans.* $y = \text{C. F.} - \frac{2}{3} \sin 2x.$

17. $2y'' - 3y' + y = 3 \cos x - 2e^{-2x} + 5.$

Ans. $y = \text{C. F.} - \frac{3}{10} \cos x - \frac{9}{10} \sin x - \frac{2}{15} e^{-2x} + 5.$

18. $6y'' - 5y' + y = 3e^{\frac{1}{2}x} - \sin 2x.$

Ans. $y = \text{C. F.} + 3xe^{\frac{1}{2}x} + \frac{23}{629} \sin 2x - \frac{10}{629} \cos 2x.$

19. $y'' - 2y' + y = xe^x.$ *Ans.* $y = e^x(c_1 + c_2 x + \frac{1}{6} x^3).$

20. $\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2 - t + 2te^t.$

Ans. $y = \text{C. F.} - t + \frac{1}{2} t^2 - 2te^t + t^2 e^t.$

In the following cases, find a particular solution satisfying the given conditions.

21. $y'' = 4 - y;\ y = y' = 0$ when $x = 0.$ *Ans.* $y = 4(1 - \cos x).$

22. $\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2t;\ x = 0$ when $t = 0,\ x = -1$ when $t = 1.$

Ans. $(1 - e)(x + 2t + t^2) = 2(1 - e^t).$

23. $y'' - 3y' + 2y = 2e^{3x};\ y = y' = 0$ when $x = 0.$

Ans. $y = e^x - 2e^{2x} + e^{3x}.$

24. $y'' + y = 3 \sin x;\ y = 1$ when $x = 0,\ y = 0$ when $x = \frac{1}{2}\pi.$

Ans. $y = (1 - \frac{3}{2} x) \cos x.$

25. $y'' - 4y' + 3y = x;\ y = -2,\ y' = 1$ when $x = 0.$

CHAPTER XXVII

RECTILINEAR MOTION

199. Introduction. Consider a point P moving in a straight line: for instance, the centroid of a falling body, of the piston of a steam engine, or of a train running on a straight track. The position of the point at any instant is determined by its abscissa $OP = x$, counted from an arbitrarily chosen origin O on the line, a definite sense along the line being selected as positive.

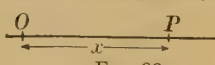


FIG. 99

In § 66 we have considered the case in which the abscissa x is given as a function of the time:

$$x = \phi(t).$$

If this function is known, the motion of the point is completely determined. The velocity v is found as the first derivative, and the acceleration j as the second derivative, of the abscissa x with respect to the time.

In practice, however, it seldom happens that the distance x is known in terms of t. The problem of greatest practical importance, and the one to be studied in this chapter, is that in which the acceleration is given, as a function of t, x, and v (or of any one or two of these), say

$$\frac{d^2x}{dt^2} = f(t, x, v). \tag{1}$$

The abscissa x is found in terms of t by *solving this differential equation of the second order.*

It should be noted that when the acceleration is given, the motion is not completely determined unless "initial conditions" are also given by means of which the constants of integration can be determined.

200. Motion of a particle under given forces. Suppose the "point" whose motion was discussed in the preceding article is a material particle moving under given forces. If the particle* is free to move in any direction, the motion will be rectilinear only if the resultant F of all the applied forces lies in the same straight line with the initial velocity. The product of the mass by the acceleration is equal to the resultant force (§ 123). If we multiply both members of equation (1) above by m, and write $F(t, x, v)$ in place of $mf(t, x, v)$, that equation takes the form

$$m\frac{d^2x}{dt^2} = F(t, x, v).$$

This equation and equation (1) of § 199 are mathematically equivalent, since one is a mere constant multiple of the other. The difference lies in the physical meaning of the quantities involved.

201. The equation of motion. The equation

$$\frac{d^2x}{dt^2} = f(t, x, v), \tag{1}$$

or its equivalent

$$m\frac{d^2x}{dt^2} = F(t, x, v), \tag{2}$$

is called the *equation of motion.* It follows from what has been said that the rectilinear motion of a particle is determined by the equation of motion together with the initial conditions.

In each problem there are in general three steps: first, *to write the equation of motion;* second, *to solve this equation,* determining the constants of integration in accordance with given initial conditions; third, *to interpret the results.*

* It should be noted that the term "particle" as here used does not mean necessarily a mere mass-point. The "particle" may be a body of any size or shape, provided that all the forces acting may be regarded as applied at a single point, and that the motion of one point determines the motion of the whole mass, as in the case of a rigid body moving without rotation.

When the forces acting are given, the equation of motion can be written at once: we have only to *equate* $m\frac{d^2x}{dt^2}$ *to the sum of the components of the forces in the direction of motion.*

In any problem we may desire to know the position of the particle at any time, the velocity at any time, and the velocity at any position. For a complete discussion, therefore, we should try to obtain three equations, giving x in terms of t, v in terms of t, and v in terms of x, respectively.

Throughout this and the following chapter, distance will be measured in feet and time in seconds, except where the contrary is indicated.

202. Uniformly accelerated motion. In uniformly accelerated motion (cf. § 66) the equation of motion is evidently

$$\frac{d^2x}{dt^2} = k,$$

where k is the constant acceleration.

Example: A body moves in a vertical line under the acceleration of gravity; the initial velocity is 60 ft. per sec. upward. Study the motion completely.

Taking the starting point as origin, and the downward sense as positive*, we have to solve the equation

$$\frac{d^2x}{dt^2} = g \tag{1}$$

subject to the conditions $x = 0$, $v = -60$ when $t = 0$. Integrating once (§ 187), we find, with $g = 32$,

$$v = 32\,t + c_1.$$

The initial conditions give

* This choice is, of course, arbitrary. With the upward sense as positive, we would have

$$\frac{d^2x}{dt^2} = -g, \text{ with } x = 0 \text{ and } v = 60 \text{ when } t = 0,$$

leading to the same conclusions as obtained in the text.

$$-60 = 0 + c_1,$$

whence

(2) $$v = 32\,t - 60.$$

A second integration gives

$$x = 16\,t^2 - 60\,t + c_2,$$

or after determination of the constant,

(3) $$x = 16\,t^2 - 60\,t.$$

Writing (1) in the form (§ 189)

$$v\frac{dv}{dx} = 32,$$

we get

$$v^2 = 64\,x + c_1',$$

which becomes as a result of the initial conditions

(4) $$v^2 = 64\,x + 3600.$$

Setting $v = 0$ in (2), we find that the body rises during a time $t = \frac{60}{32} = 1\frac{7}{8}$ sec.; setting $v = 0$ in (4), we find that the highest point is $\frac{3600}{64} = 56\frac{1}{4}$ ft. above the starting point. After reaching the highest point the body falls with continually increasing velocity.

203. Atwood's machine. In the apparatus called *Atwood's machine*, two masses m_1, m_2 are joined by a cord hung over a pulley, as in Fig. 100. Suppose for definiteness that $m_1 > m_2$.

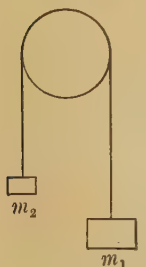


FIG. 100

The total mass moved (if the masses of the cord and pulley can be neglected) is

$$m = m_1 + m_2,$$

while the force producing the motion is

$$F = m_1 g - m_2 g.$$

Hence equation (2) of § 201 becomes

$$(m_1 + m_2)\frac{d^2x}{dt^2} = (m_1 - m_2)g,$$

or

$$\frac{d^2x}{dt^2} = \frac{m_1 - m_2}{m_1 + m_2}\,g.$$

EXERCISES

Study the motion of a body in a vertical line under gravity, under the following conditions. Discuss the results fully.

1. The initial velocity is 12 ft. per sec. upward.
Ans. $v = 32\,t - 12$, $x = 16\,t^2 - 12\,t$, $v^2 = 64\,x + 144$.

2. The initial velocity is 6 ft. per sec. downward.
Ans. $v = 32\,t + 6$, $x = 16\,t^2 + 6\,t$, $v^2 = 64\,x + 36$.

3. The body rises $\frac{3}{4}$ sec. before starting downward.

4. The velocity after 2 sec. is 2 ft. per sec. downward.

5. Given $x = 2$ when $t = 1$ and $x = 6$ when $t = 2$.
Ans. $x = 16\,t^2 - 44\,t + 30$, $v = 32\,t - 44$, $v^2 = 64\,x + 16$.

6. The velocity 4 ft. above the starting point is 3 ft. per sec.

7. The velocity 2 ft. below the starting point is 23 ft. per sec. If the starting point is 500 ft. above the earth's surface, when and with what velocity does the body reach the earth? *Ans.* $t = 5$ or $6\frac{1}{4}$ sec.

8. After 2 sec. the body is 4 ft. above the starting point.

9. The body rises 160 ft. before starting to fall.

10. The body falls 50 ft. in the third second of motion.
Ans. $v = 32\,t - 30$, etc.

11. In the first 100 ft. of the fall the velocity increases by 40 ft. per sec.

12. If a stone dropped from a balloon while ascending at the rate of 20 ft. per sec. reaches the ground in 10 sec., what was the height of the balloon when the stone was dropped? With what velocity does the stone strike the ground? *Ans.* 1400 ft.; 300 ft. per sec.

13. A body falls under gravity. Find the distance covered in 6 sec. if at the end of 2 sec. the distance below the starting point is 84 ft. *Ans.* 636 ft.

14. A stone is thrown vertically upward from the top of a tower. At the end of 2 sec. it is 400 ft. above the ground, and is still rising, with a velocity of 10 ft. per sec. Find the height of the tower, and the time of flight. *Ans.* 316 ft.; 7.3 sec.

15. A stone thrown upward from the top of a tower with a velocity of 100 ft. per sec. reaches the ground with a velocity of 140 ft. per sec. Discuss the motion. Find the height of the tower, and the time of flight. *Ans.* 150 ft.; 7.5 sec.

16. A stone is thrown upward from the top of a tower. At the end of 2 sec. it is 84 ft., at the end of 3 sec. 36 ft., above the ground. Find the height of the tower, the time of flight, and the initial and final velocities. *Ans.* Height = 84 ft.

17. If a body slides down a smooth inclined plane*, the "effective component" of gravity — *i.e.* the component in the direction of the motion — is evidently $g \sin \alpha$, where α is the angle of inclination of the plane to the horizontal. Study the motion of a body sliding from rest on a plane inclined at 45°.

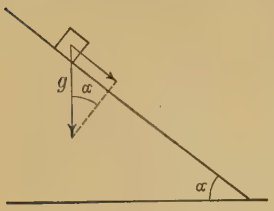


FIG. 101

Ans. $v = 16\sqrt{2}\, t$, $x = 8\sqrt{2}\, t^2$, $v^2 = 32\sqrt{2}\, x$.

18. Study the motion of a body on a plane inclined at 30° to the horizontal, if the initial velocity is 10 ft. per sec. up the slope.

19. Study the motion of a body on a plane inclined at 30° to the horizontal, if the body rises 20 ft. (measured along the slope) before starting to fall.

Ans. $v^2 = 32\, x + 640$, $v = 16\, t - 25.3$, $x = 8\, t^2 - 25.3\, t$.

20. A body moves on an inclined plane. After 1 sec. the body is 6 ft. below the starting point, and has a velocity of 10 ft. per sec. Find the inclination of the plane. *Ans.* About 14° 30′.

21. A bead is strung on a smooth straight wire inclined at 45° to the horizontal. What initial velocity must the bead be given to raise it to a vertical height of 10 ft.? *Ans.* 25.3 ft. per sec.

22. A railroad train is running up a grade of 1 in 200 at the rate of 20 mi. per hr. when the coupling of the last car breaks. Friction being neglected, (*a*) how far will the car have gone after 2 min. from the point where the break occurred? (*b*) When will it begin moving down the grade? (*c*) How far will it be behind the train at that moment? (*d*) If the grade extends 1500 ft. below the point where the break occurred, with what velocity will it reach the foot of the grade?

Ans. (*a*) 2368 ft.; (*b*) 3 min. 3 sec.; (*c*) 2689 ft.; (*d*) 25 mi. per hr.

23. Show that it takes a body twice as long to slide down a plane of 30° inclination as it would take to fall through the "height" of the plane.

24. Show that in sliding down a smooth inclined plane a body acquires the same velocity as in falling vertically through the height of the plane.

25. What uniform acceleration will bring an automobile, running at 40 mi. per hr., to rest in 100 ft.? What time will be required?

Ans. $-$ 17.2 ft. per sec^2.; 3.4 sec.

* In Exs. 17–20, it is understood that the motion takes place along a line of greatest slope — *i.e.* a line at right angles to a horizontal line in the plane.

26. The motion of a railroad train is uniformly accelerated. If when the train is 250 ft. from a station the velocity is 30 ft. per sec., when 600 ft. from the station it is 40 ft. per sec., find the acceleration, and the velocity when passing the station. *Ans.* $v_0 = 20$ ft. per sec.

27. A cord hangs over a vertical pulley and carries equal weights of 10 lbs. at each end. If a 1-lb. weight be added at one end, discuss the motion of the system. Find v when the system has moved 6 ft.

Ans. 4.3 ft. per sec.

28. The weights in Atwood's machine are 6 and 10 lbs. If the smaller weight is originally falling 4 ft. per sec., discuss the motion.

29. The weights in Atwood's machine are 4 and 10 lbs.; the cord is 3 ft. long. If the weights are initially equidistant from the pulley, what velocity must be given the system to make the heavier weight strike the pulley? *Ans.* 6.4 ft. per sec.

30. A mass of 12 lbs. rests on a smooth horizontal table. A cord attached to this mass runs over a pulley on the edge of the table; from the cord a mass of 4 lbs. is suspended. Discuss the motion. If the 12-lb. mass is originally 5 ft. from the edge of the table, find when and with what velocity it reaches the edge. *Ans.* 1.1 sec.; 8.9 ft. per sec.

31. A car, starting with a velocity of 10 mi. per hr., coasts for 20 sec. down a 2% grade (2 vertical in 100 horizontal) and then ascends a 3% grade. Neglecting friction, find how far it will go up the grade.

Ans. 393 ft.

32. A car, starting from rest, coasts 200 ft. down a 2% grade and then ascends a 10% grade under an acceleration, due to its own power, of 2 ft. per sec. per sec. How far up can it go? *Ans.* 107 ft.

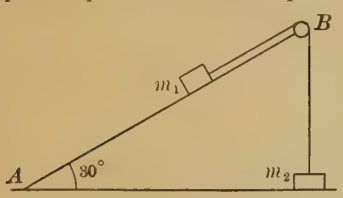


FIG. 102

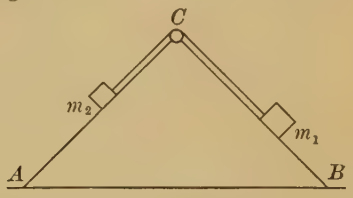


FIG. 103

33. A mass $m_1 = 6$ lbs. rests on an inclined plane and is attached to a mass $m_2 = 2$ lbs. by a cord hung over a pulley, as shown in Fig. 102. If $AB = 20$ ft., and m_2 is originally at rest on the ground, when and with what velocity does m_2 strike the pulley? *Ans.* 2.2 sec.; 8.9 ft. per sec.

34. In Fig. 102, if the cord is 25 ft. long and m_2 is on the ground, what velocity must be given the system to make m_2 strike the pulley?

Ans. 16.7 ft. per sec.

35. In Fig. 103, $AC = BC = 25$ ft., $m_1 = 15$ lbs., $m_2 = 10$ lbs., the cord is 26 ft. long. If m_1 is originally at rest 15 ft. from C, find when and with what velocity it strikes the ground.

Ans. 2.1 sec.; 9.5 ft. per sec.

36. In Ex. 35, with what velocity does m_2 strike the pulley?

Ans. 6.7 ft. per sec.

Study the following motions. In each case, examine v and x for maxima and minima. Discuss fully.

37. $j = 6t - 4$; $x = 0$, $v = 1$ when $t = 0$. *Ans.* $x = t^3 - 2t^2 + t$.

38. $j = 8t - 4$; $x = 0$, $v = 1$ when $t = 0$.

Ans. $x = \frac{4}{3}t^3 - 2t^2 + t$.

39. $j = 2t - 2$; $x = 0$, $v = 2$ when $t = 0$.

Ans. $x = \frac{1}{3}t^3 - t^2 + 2t$.

40. $j = 6 - 2t$; $x = 0$, $v = -5$ when $t = 0$.

41. $j = 3t^2 - 8t + 5$; $x = 0$ when $t = 0$, $x = -\frac{7}{12}$ when $t = 1$.

42. $j = 3t^2 - 4t + 1$; $x = v = 0$ when $t = 0$.

43. $j = t^2 - 3t + 2$; $x = 0$, $v = 1$ when $t = 0$.

44. $j = 3(t-1)^2$; $x = v = 0$ when $t = 0$.

45. $j = 3t^2 - 12t + 11$; $x = 0$, $v = -6$ when $t = 0$.

46. $j = \dfrac{1}{t+1}$; $x = 0$, $v = -2$ when $t = 0$.

47. $j = 1 + \dfrac{2}{(t+1)^2}$; $x = 0$, $v = -1$ when $t = 0$.

48. $j = 1 - \dfrac{20}{(t+2)^2}$; $x = 0$, $v = 3$ when $t = 0$.

49. $j = 2 \sin 3t$; $x = v = 0$ when $t = 0$.

50. $j = 4 \sin \frac{1}{2}t$; $x = 0$, $v = -8$ when $t = 0$.

51. $j = \sin 4t$; $x = 0$ when $t = 0$; $x = \frac{1}{8}\pi$ when $t = \pi$.

52. $j = -4 \sin \frac{1}{4}t$; $x = 0$, $v = 32$ when $t = 0$.

53. $j = 2 \sin 2t$; $x = 0$, $v = 1$ when $t = 0$.

54. $j = 3.72 \cos (t + 0.27)$; $x = v = 0$ when $t = 0$.

Ans. $x = 3.58 - 0.99t - 3.72 \cos (t + 0.27)$.

55. $j = g - 32 \cos 4t$; $x = 0$, $v = -4$ when $t = 0$.

204. Motion in a resisting medium. It is a matter of common observation that, when a body moves in a fluid medium, such as air or water, the medium offers a *resistance* to the motion. Experiment shows that the magnitude of this resistance depends upon the velocity, and in fact that it increases as the velocity increases; but the law of variation

is not known. However, valuable approximate results are obtained in some cases by assuming the resistance proportional to the velocity; in other cases a better approximation may be obtained by assuming the resistance proportional to the square of the velocity.

In each problem, the sign to be given to the resistance must be determined from the fact that *the resistance acts always in the direction opposite to the motion.*

Example: A body falls from rest under gravity in a medium offering a resistance proportional to the velocity. Study the motion.

With the downward sense chosen as positive, the equation of motion is

$$\frac{d^2x}{dt^2} = g - kv, \tag{1}$$

the negative sign being chosen because when the body is falling (v positive), the resistance is directed upward (negative). Replacing $\frac{d^2x}{dt^2}$ by $\frac{dv}{dt}$ (§ 188) and separating variables, we get

$$\frac{dv}{g - kv} = dt,$$

whence

$$\log (g - kv) = - kt + c_1.$$

Since $v = 0$ when $t = 0$ this becomes

$$\log (g - kv) = - kt + \log g,$$

or

$$v = \frac{g}{k}(1 - e^{-kt}). \tag{2}$$

Replacing v by $\frac{dx}{dt}$ and integrating again, we find

$$x = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + c_2,$$

or finally, since $x = 0$ when $t = 0$,

$$x = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2}.$$

EXERCISES

1. Study the motion of a body in a medium offering a resistance proportional to the velocity, with no other forces acting. Solve the equation of motion in three ways (§§ 188, 189, 193). Interpret the results fully *. Assume $x = 0$, $v = v_0$ when $t = 0$.

Ans. $x = \frac{v_0}{k}(1 - e^{-kt})$, $v = v_0 e^{-kt}$, $v = v_0 - kx$.

2. In Ex. 1, draw the graphs of x, v, and j as functions of t, taking $k = 0.5$, $v_0 = 1$.

3. A body is placed on a smooth horizontal table and given a velocity of 2 ft. per sec. After 2 sec. the velocity is observed to be 1.82 ft. per sec. Assuming the resistance of the air as proportional to v, find k. *Ans.* 0.047.

4. Under the conditions of Ex. 3, find k if the body comes to rest 11.36 ft. from the starting point. *Ans.* 0.176.

5. In the example (§ 204), discuss the results, noting particularly the nature of the motion after a long time.

6. Solve the example of § 204 by the method of § 198.

7. In the example of § 204, draw the graphs of x, v, and j as functions of t, assuming $k = 1$. (Cf. § 38.)

8. In the example of § 204, find the distance traveled in the first 10 seconds, (*a*) if $k = 1$; (*b*) if $k = 0.01$. Compare with the case of negligible resistance. *Ans.* (*a*) 288 ft.; (*b*) 1548 ft.

9. An aeronaut makes a parachute drop from a height of 5000 ft. Assuming $k = 1$ in § 204, find the time of fall. Can this drop be made with safety? *Ans.* 2 min. 37 sec.

10. A body falling from rest in a heavy fluid acquires in 2 sec. a velocity of 9.33 ft. per sec. Assuming the resistance of the medium to be proportional to the velocity, find k. *Ans.* 3.4.

11. A body falling from rest in a heavy fluid travels 1.68 ft. in the first second. Assuming the resistance of the medium to be proportional to the velocity, find k. *Ans.* 18.0.

12. Study the motion of a body in a medium offering a resistance proportional to the square of the velocity, with no other forces acting.

Ans. $v = \frac{v_0}{1 + kv_0 t}$, $x = \frac{1}{k}\log(1 + kv_0 t)$, $v = v_0 e^{-kx}$.

* Strictly interpreted, the solution shows that the body would never come to rest, a result which is not in accord with observation. The reason that theory and practice do not fully agree in this and similar problems is that the original assumption is only an approximation.

13. In Ex. 12, draw the graphs of x, v, and j as functions of t, taking $k = 0.2$, $v_0 = 10$.

14. Solve Ex. 3, assuming the resistance to be proportional to v^2.

Ans. $k = 0.025$.

15. A body is projected upward with a velocity of 1000 ft. per sec. Taking the resistance of the air as proportional to the velocity, with $k = 0.01$, find how far and for how long a time the body will rise.

Ans. 12,960 ft.; 27.2 sec.

16. A body is thrown upward with a velocity of 80 ft. per sec. in a medium offering a resistance proportional to v^2. Taking $k = 0.005$, find how far and for how long a time the body will rise.

Ans. 69.3 ft., 1.96 sec.

17. In Ex. 16, find when and with what velocity the body returns to the starting point. *Ans.* 2.2 sec.; 56.6 ft. per sec.

18. A body is thrown downward with a velocity v_0. Taking the resistance of the air as proportional to v, study the motion. Distinguish the cases $v_0 \neq \frac{g}{k}$, $v_0 = \frac{g}{k}$.

19. A body thrown downward with a velocity of 117.3 ft. per sec. is observed to have approximately uniform motion from the beginning. Assuming the resistance of the air to be proportional to v, find k. (Ex. 18.) *Ans.* 0.27.

20. Study the upward motion in Ex. 18, p. 336, considering the resistance of the medium to be proportional to v^2. Take $k = 0.0025$.

Ans. $v^2 = 6500\, e^{-0.005x} - 6400$, $v = 80 \tan (0.2\, t - 0.124)$,

$$x = -0.59 - 400 \log \cos (0.2\, t - 0.124).$$

21. In Ex. 20, study the downward motion.

Ans. $v^2 = 6400(1 - e^{-0.005x})$, $v = 80\, \frac{e^{0.4t} - 1}{e^{0.4t} + 1} = 80 \tanh 0.2\, t$,

$$x = 400 \log \cosh 0.2\, t.$$

22. Study the motion of a body under a force proportional to the velocity and *assisting* the motion. Take $x = 0$, $v = v_0$ when $t = 0$.

23. A body moves in a vertical line under gravity, and under the action of a force proportional to the velocity and assisting the motion. Assuming $x = v = 0$ when $t = 0$, study the motion. Take $k = 4$.

Ans. With the upward sense as positive, $x = 8\, t + 2 - 2\, e^{4t}$, etc.

24. Solve Ex. 23 if there is an upward initial velocity greater than 8 ft. per sec.

25. Solve Ex. 23 if there is an upward initial velocity of 8 ft. per sec.

26. Solve Ex. 23 if there is an upward initial velocity of 3 ft. per sec. For what time, and through what distance, does the body rise?

Ans. $t = 0.12$ sec., $x = 0.19$ ft.

27. Investigate the motion

$$j = -kv + pt,$$

with $x = v = 0$ when $t = 0$. *Ans.* $x = \frac{p}{k^3}(1 - e^{-kt}) - \frac{p}{k^2}t + \frac{p}{2k}t^2$.

28. Solve Ex. 27 with $x = 0$, $v = \frac{p}{k^2}$ when $t = 0$.

29. Investigate the motion

$$j = g - kv + pt,$$

taking $x = v = 0$ when $t = 0$.

Ans. $x = \left(\frac{p}{k^3} - \frac{g}{k^2}\right)(1 - e^{-kt}) - \left(\frac{p}{k^2} - \frac{g}{k}\right)t + \frac{p}{2k}t^2$.

205. Simple harmonic motion. If a particle moves in a straight line under the action of a force directed toward a fixed point O in the line of motion, and proportional to the distance x from that point, it is said to have *simple harmonic motion.* The equation of motion can evidently be written in the form

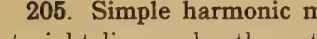


FIG. 104

$$\frac{d^2x}{dt^2} = -k^2x, \tag{1}$$

where k is a constant, the minus sign being chosen because the force is always directed opposite to the "displacement" x. The fixed point toward which the force is directed is called the *center of force.*

Integrating equation (1) by the method of § 196, we get

$$x = c_1 \cos kt + c_2 \sin kt,$$

whence

$$v = \frac{dx}{dt} = -kc_1 \sin kt + kc_2 \cos kt.$$

Take $v = 0$ and $x = a$ when $t = 0$. Then

$$a = c_1, \quad 0 = kc_2,$$

whence

$$x = a \cos kt, \tag{2}$$

$$v = -ak \sin kt. \tag{3}$$

Since the cosine is a periodic function, it follows from (2) that the body moves forever back and forth between the points $x = a$ and $x = -a$ (the points A, B in the figure), while (3) shows that v varies between the extreme values ak and $-ak$.

The time required for a complete oscillation, from A to B and back, is called the *period* of the motion. When $x = a$, $\cos kt = 1$, and the successive values of t are $0, \frac{2\pi}{k}, \frac{4\pi}{k}$, etc.; hence the period is $\frac{2\pi}{k}$. The extreme distance a is called the *amplitude*.

206. Hooke's law. When a spiral steel spring of length $AO = l$ is stretched to a length $AP = l + x$, the tension in the spring, or the force tending to restore it to its natural length, is *proportional to the extension* x. This law, known as *Hooke's law*, is obeyed very closely (provided the extension is not too great) by all so-called *elastic* materials.

Suppose a steel spring of negligible mass is placed on a smooth horizontal table with one end fast at A. Let the natural length of the spring be $AO = l$. A particle of mass m attached to the free end is drawn out to the position P and then released. The only force acting is the tension in the spring, which by Hooke's law is directed toward the position of equilibrium O and is proportional to the distance from O. If the spring offers the same resistance to compression as to extension, it follows from § 205 that the particle performs simple harmonic oscillations about O, the equation of motion being

FIG. 105

$$\frac{d^2x}{dt^2} = -k^2x.$$

Of course if the resistance to compression is not the same as to extension, a different equation comes into play as soon as the particle passes through O.

EXERCISES

1. In § 205, find the (v, x) equation. *Ans.* $v = \pm k\sqrt{a^2 - x^2}$.

2. In § 205, draw the graphs of x, v, and j as functions of t, taking $a = \frac{1}{2}$, $k = 2$.

3. Solve the problem of simple harmonic motion, taking $v = v_0$ and $x = 0$ when $t = 0$.

4. In Ex. 3, draw the graphs of x, v, and j, taking $a = 2$, $k = 0.2$.

5. A steel spring offering the same resistance to compression as to extension is placed on a smooth horizontal table with one end fixed. The spring is stretched to a length 6 in. greater than the natural length and then released. Discuss the motion of a mass attached to the free end. Take $k^2 = 4$. Find the period. *Ans.* Period = 3.14 sec.

6. Work Ex. 5 if the resistance to compression is four times as great as to extension. *Ans.* Period = 2.36 sec.

7. Work Ex. 5 if the steel spring is replaced by a rubber band of natural length 1 ft. *Ans.* Period = 7.14 sec.

8. In Ex. 5 discuss the effect of increasing k.

9. A body falls from rest under gravity, and under the action of a force directed toward the starting point and proportional to the distance from that point. Show that the motion is simple harmonic, and find the center. *Ans.* $x = \frac{g}{k^2}(1 - \cos kt)$.

FIG. 106

10. A rubber band of natural length $AB = l$ is suspended vertically with a weight attached. The effect of the weight is to stretch the band to a length $AO = l + h$. The weight is given a displacement $OP = a (a < h)$ and then released. Write the equation of motion with O as origin, determining k^2 from the fact that at O the acceleration is 0. What is the nature of the motion? *Ans.* $\frac{d^2x}{dt^2} = -\frac{g}{h}x$.

11. In Ex. 10, the weight is let fall from a height b above B. What is the greatest extension? *Ans.* $h + \sqrt{h^2 + 2bh}$.

12. Solve the problem of simple harmonic motion with $k^2 = 4$, taking $x = 1$, $v = 2$ when $t = 0$. *Ans.* $x = \sqrt{2}\cos(2t - \frac{1}{4}\pi)$.

13. Discuss the simple harmonic motion for which $k^2 = 4$, if $x = 2$, $v = -3$ when $t = 0$. *Ans.* $x = \frac{5}{2}\cos(2t + 0.64)$.

14. Discuss the simple harmonic motion for which $k^2 = 9$, if $x = -3$, $v = 1$ when $t = 0$. *Ans.* $x = -3.02\cos(3t + 0.11)$.

15. Solve the problem of simple harmonic motion with $x = a$, $v = v_0$ when $t = 0$. Find the amplitude and the period.

Ans. $x = \frac{1}{k}\sqrt{v_0^2 + a^2k^2}\cos(kt - \epsilon)$, where $\epsilon = \arctan\frac{v_0}{ak}$.

16. It is shown in the theory of attraction that the attraction of a spherical mass on a particle within the mass is directed toward the center of the sphere and is proportional to the distance from the center. Discuss the motion of a particle moving in a straight tube through the center of the earth, if the velocity at the surface is 0. Determine the proportionality constant k^2 from the fact that the force at the surface is $-mg$. How long does it take the particle to pass through the earth?

Ans. $42\frac{1}{2}$ minutes.

17. A straight tube is bored through the earth connecting two points of its surface. Show that the equation of motion of a particle sliding in this tube is

$$m\frac{d^2x}{dt^2} = -\frac{mg}{R}x,$$

where R is the radius of the earth and x is the distance of the particle from the midpoint of the tube. Discuss the motion. Show that the period is independent of the position of the endpoints.

18. A particle is acted upon by a force of *repulsion* from a point O proportional to the distance from O. Write the equation of motion and solve it completely, taking $x = 0$ and $v = v_0$ when $t = 0$. Describe the motion.

Ans. $x = \frac{v_0}{k}\sinh kt$, etc.

19. A body is acted upon by a force of repulsion from a point O proportional to the distance from O. If the motion takes place in a vertical line under gravity, with $x = v = 0$ when $t = 0$, discuss the motion fully. Assume $k^2 = 4$.

Ans. With the upward sense as positive, $x = 8(1 - \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t})$.

20. Solve Ex. 19 if there is an upward initial velocity of 20 ft. per sec.

21. Solve Ex. 19 if there is an upward initial velocity of 10 ft. per sec. For what time and through what distance does the body rise?

Ans. $t = 0.37$ sec., $x = 1.8$ ft.

22. Solve Ex. 19 if there is an upward initial velocity of 16 ft. per sec.

23. Study the motion $j = -k^2x + pt$, with $x = v = 0$ when $t = 0$.

Ans. $x = -\frac{p}{k^3}\sin kt + \frac{p}{k^2}t$.

24. Solve Ex. 23 with $x = 0$, $v = \frac{p}{k^2}$ when $t = 0$.

25. Study the motion $j = -k^2x + pt^2$, with $x = v = 0$ when $t = 0$.

Ans. $x = \frac{2p}{k^4}\cos kt + \frac{p}{k^2}t^2 - \frac{2p}{k^4}$.

207. Damped vibrations. We will consider next the case in which the body moves in a medium offering a resistance proportional to the velocity, and under an attraction proportional to the displacement. The equation of motion is evidently

$$\frac{d^2x}{dt^2} = -k^2x - k'v,$$

or, if we put for convenience $k' = 2h$,

$$\frac{d^2x}{dt^2} + 2h\frac{dx}{dt} + k^2x = 0. \tag{1}$$

The characteristic equation (§ 192) is

$$m^2 + 2hm + k^2 = 0,$$

whence

$$m = -h \pm \sqrt{h^2 - k^2}.$$

The case of greatest interest is that in which $h < k$, so that the roots are imaginary. Setting

$$\beta = \sqrt{k^2 - h^2},$$

we have

$$m = -h \pm \beta i,$$

so that the general solution of (1) is

$$x = e^{-ht}(c_1 \cos \beta t + c_2 \sin \beta t).$$

With the initial conditions $x = 0$, $v = v_0$ when $t = 0$, we find (the details are left to the student)

$$x = \frac{v_0}{\beta} e^{-ht} \sin \beta t, \tag{2}$$

$$v = \frac{v_0}{\beta} e^{-ht}(\beta \cos \beta t - h \sin \beta t). \tag{3}$$

Thus v vanishes whenever

$$\tan \beta t = \frac{\beta}{h}, \quad \beta t = \arctan \frac{\beta}{h} + n\pi,$$

i.e. when

$$t = \frac{1}{\beta} \arctan \frac{\beta}{h} + \frac{n\pi}{\beta}. \tag{4}$$

The displacement x therefore has an infinite number of maxima and minima whose absolute values steadily diminish

(see Ex. 2 below); that is, the body vibrates back and forth indefinitely, but the resistance of the medium continually shortens the amplitude.

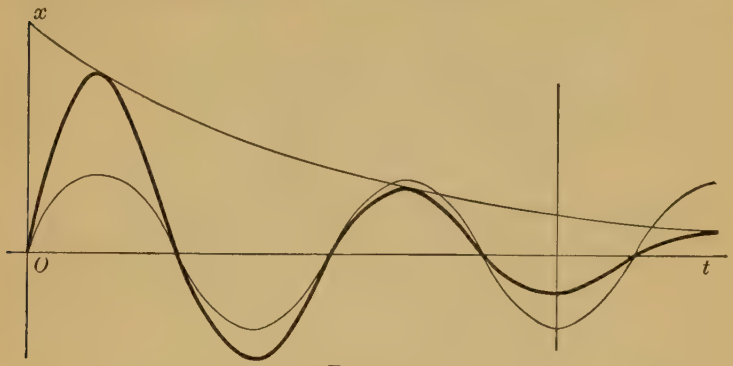


FIG. 107

Figure 107 shows the graph of x as given by (2), found by multiplying the ordinates of the curves

$$x = \frac{v_0}{\beta} e^{-ht}, \quad x = \sin \beta t.$$

EXERCISES

1. Derive equations (2) and (3) above.

2. By substituting the value of t from (4) in (2), find the extreme values of x, and obtain the ratio of two consecutive amplitudes.

Ans. The ratio is (numerically) $e^{-\frac{h\pi}{\beta}}$.

3. Show that the period of the damped vibrations is constant, and greater than in the case of "free" oscillations (§ 205).

Ans. Period $= \dfrac{2\pi}{\sqrt{k^2 - h^2}}$.

4. Find when v assumes its maximum and minimum values.

Ans. $t = \dfrac{1}{\beta} \arctan \dfrac{2h\beta}{h^2 - \beta^2} + \dfrac{n\pi}{\beta}$.

5. How is the motion affected (*a*) if h is very small; (*b*) if h is but slightly less than k?

6. In Fig. 107, do the maximum and minimum points on the product curve coincide with its points of intersection with the exponential curve?

7. Solve the case $h < k$ with the initial conditions $x = a, v = 0$ when $t = 0$. *Ans.* $x = \frac{a}{\beta} e^{-ht}(\beta \cos \beta t + h \sin \beta t)$, $v = -\frac{ak^2}{\beta} e^{-ht} \sin \beta t$.

8. Solve equation (1), § 207, with $h = 5$, $k = 4$, taking $x = 12$, $v = 0$ when $t = 0$. Describe the motion.

Ans. $x = 16\, e^{-2t} - 4\, e^{-8t}$, etc.

9. In Ex. 8, draw the graph of x.

10. Solve Ex. 8, taking $x = 0$, $v = v_0$ when $t = 0$. Show that the time of reaching the maximum displacement is independent of the initial velocity. *Ans.* $x = \frac{1}{6} v_0(e^{-2t} - e^{-8t})$.

11. Solve Ex. 8 if $x = 12$, $v = -300$ when $t = 0$. Find when the body passes through O, and when it reaches the maximum negative displacement. *Ans.* $t = 0.05$ sec.; $t = 0.28$ sec.

12. Discuss fully the case $h = k$, taking $x = 0$, $v = v_0$ when $t = 0$. Find the maximum displacement. *Ans.* $x = v_0 t e^{-ht}$.

13. In Ex. 12, draw the graph of x.

14. Solve the case $h = k$ if $x = a, v = 0$ when $t = 0$. Find the maximum velocity. *Ans.* $x = ae^{-kt}(1 + kt)$.

15. Discuss the case $h = k$, taking $x = -a$, $v = v_0$ when $t = 0$. Distinguish the cases $v_0 < ah$, $v_0 = ah$, $v_0 > ah$.

16. Solve the case $h = k = 1$ if $x = 1$ when $t = 0$, $x = 0$ when $t = 1$.

Ans. $x = e^{-t}(1 - t)$.

17. Solve the case $h = 1, k = \sqrt{2}$, taking $x = 1, v = -1$ when $t = 0$.

Ans. $x = e^{-t} \cos t$.

18. In Ex. 17, draw the graph of x as a function of t.

19. Solve the case $h = 1$, $k = \sqrt{2}$, taking $x = 1$ when $t = 0$, $x = 2$ when $t = 1$. *Ans.* $x = 5.88\, e^{-t} \cos (t - 1.40)$.

20. Solve the case $h = 1$, $k = \sqrt{2}$ if $v = 0$ when $t = 0$, $v = 1$ when $t = \frac{1}{2}\pi$. *Ans.* $x = -3.41\, e^{-t} \cos (t - \frac{1}{4}\pi)$.

21. Solve the case $h = 1$, $k = \sqrt{2}$, taking $x = 1$, $v = 10$ when $t = 0$. Find the maximum displacement.

Ans. $x = 11.05\, e^{-t} \cos (t - 1.48)$; $x_{\max.} = 3.9$.

22. Solve the case $h = k$ if the motion takes place in a vertical line under gravity. Assume $x = v = 0$ when $t = 0$. Compare with the case of negligible resistance (Ex. 9, p. 344).

Ans. $x = \frac{g}{k^2}(1 - e^{-kt} - kte^{-kt})$.

23. Solve Ex. 22 if $h = k = \frac{1}{2}$, assuming $x = 0$ and $v = 64$ ft. per sec. upward when $t = 0$. Find the highest and lowest points.

Ans. $x = 128(1 - e^{-\frac{1}{2}t} - te^{-\frac{1}{2}t})$; 27.3 ft., 128 ft.

24. Solve the problem of damped vibrations in a vertical line under gravity, taking $h = \frac{1}{6}$, $k^2 = \frac{1}{48}$, and the initial velocity 100 ft. per sec. upward. For how long a time does the body rise?

$$Ans.\ x = 1536 + 1368\, e^{-\frac{1}{4}t} - 2904\, e^{-\frac{1}{12}t};\ 2.08 \text{ sec.}$$

25. Solve the problem of damped vibrations in a vertical line under gravity for the case $h < k$, taking $x = v = 0$ when $t = 0$.

$$Ans.\ x = \frac{g}{k^2} - \frac{g}{k^2\beta} e^{-ht}(\beta \cos \beta t + h \sin \beta t)$$

$$= \frac{g}{k^2} - \frac{g}{k\beta} e^{-ht} \cos(\beta t - \epsilon), \text{ where } \epsilon = \arctan \frac{h}{\beta}.$$

26. Solve Ex. 19, p. 345, if the resistance of the air is proportional to the velocity, taking $x = -a$, $v = v_0$ when $t = 0$. Distinguish the cases $v_0 \gtreqless a(h + \beta)$, where $\beta = \sqrt{h^2 + k^2}$. Discuss fully.

$$Ans.\ x = c_1 e^{(-h+\beta)t} + c_2 e^{(-h-\beta)t}, \text{ where } c_1 = \frac{v_0 - a(h+\beta)}{2\beta},\ c_2 = -a - c_1.$$

27. A body moves under the action of an attraction $-mk^2x$ proportional to the displacement, and a force $2hmv$ proportional to the velocity and assisting the motion. Discuss fully the case $h < k$, taking $x = 0$, $v = v_0$ when $t = 0$. $Ans.\ x = \frac{v_0}{\beta} e^{ht} \sin \beta t$, where $\beta = \sqrt{k^2 - h^2}$.

28. In Ex. 27, trace the graph of x, taking numerical values for v_0, h, and k.

29. Solve Ex. 27 if the motion occurs in a vertical line under gravity. Take $v_0 = 0$. $Ans.\ x = \frac{g}{k^2} - \frac{g}{k^2\beta} e^{ht}(\beta \cos \beta t - h \sin \beta t)$.

30. Solve Ex. 27 in case $h = k$. $Ans.\ x = v_0 t e^{kt}$.

31. Solve Ex. 27 with $h = \frac{5}{4}$, $k = \frac{3}{4}$.

32. A body moves under the action of a repulsive force proportional to the displacement, and a force proportional to the velocity and assisting the motion. Discuss the problem fully.

33. Solve the motion

$$j = -7v - 12x + t,$$

with $x = v = 0$ when $t = 0$. $Ans.\ x = \frac{1}{5} e^{-3t} - \frac{1}{176} e^{-4t} - \frac{7}{144} + \frac{1}{12} t$.

34. Study the motion $j = -4x + 2 \cos t$, with $x = v = 0$ when $t = 0$. Draw the graph of x.

35. Study the motion $j = -1.76x + 2.36 \sin 2t$, taking $x = 1$, $v = 0$ when $t = 0$. $Ans.\ x = 1.88 \cos(1.33t - 1.01) - 1.05 \sin 2t$.

36. Study the motion $j = \cos 2t - \frac{1}{2} \cos t$, if $x = v = 0$ when $t = 0$. Draw the graph of x.

208. Momentum. The *momentum* of a moving particle is the product of the mass by the velocity:

$$M = mv.$$

If a particle moves under given forces, the momentum at any time can evidently be found from the (v, t) – equation*; the momentum at any position, from the (v, x) – equation*.

Example: In the example of § 204, find the momentum after t sec.

By equation (2) of that article, the momentum is

$$mv = \frac{mg}{k}(1 - e^{-kt}).$$

209. Impulse. When a particle moves under a *constant force* F from the time t_1 to the time t_2, the product $F(t_2 - t_1)$ of the force by the time during which it acts is called the *impulse* of the force for that time-interval.

More generally, if F varies from instant to instant, let us divide the time from t_1 to t_2 into n equal intervals Δt, multiply each Δt by the value of F at the beginning (or any other instant) of Δt, and form the sum of the products thus obtained. The limit of this sum, as Δt approaches 0, is the *impulse* of the variable force F during the interval from t_1 to t_2:

$$I = \lim_{\Delta t \to 0} \sum F \, \Delta t = \int_{t_1}^{t_2} F \, dt.$$

Example: In the example of § 204, find the impulse of the forces during the first t sec.

By (1) of that article, the total force is

$$F = mg - mkv.$$

Substituting the value of v from (2), we get

$$F = mg - mk\left(\frac{g}{k} - \frac{g}{k}e^{-kt}\right) = mge^{-kt},$$

and

$$I = mg\int_0^t e^{-kt}\,dt = \frac{mg}{k}(1 - e^{-kt}).$$

* Provided, of course, that these equations can be made to give v explicitly.

210. The principle of impulse and momentum. Let us write the equation

$$m\frac{d^2x}{dt^2} = F$$

in the form

$$m\frac{dv}{dt} = F.$$

Multiplying by dt and integrating from the time t_1, when the velocity is v_1, to the time t_2, when the velocity is v_2, we find

$$mv_2 - mv_1 = \int_{t_1}^{t_2} F\,dt. \tag{1}$$

The left member of (1) is the change of momentum in the time-interval from t_1 to t_2, the right member is the impulse of the force F. Hence we have the *principle of impulse and momentum:*

If a particle moves in a straight line, the change of momentum in any time-interval is equal to the impulse of the force during that interval.

It should be observed that what we have really done here is to find a first integral of the equation of motion by the method of § 188.

EXERCISES

1. In Ex. 1, p. 335, find the momentum (*a*) after 2 sec.; (*b*) 2 ft. above the starting point. *Ans.* (*a*) $52\,m$; (*b*) $\pm 4\,m$.

2. In Ex. 19, p. 336, find the momentum (*a*) after 1 sec.; (*b*) when the body has returned to the starting point.
Ans. (*a*) $-9.3\,m$; (*b*) $25.3\,m$.

3. In Ex. 1, p. 340, find the momentum at any time and in any position.

4. In Ex. 12, p. 340, find the momentum at any time and any place.

5. In Ex. 20, p. 341, find the momentum (*a*) after 0.5 sec.; (*b*) 2 ft. above the starting point. *Ans.* (*a*) $-1.92\,m$; (*b*) $5.92\,m$.

6. In Ex. 12, p. 344, find the momentum at any time and any place.

7. In Ex. 13, p. 344, find the momentum (*a*) when $x = 1$; (*b*) when $t = 10$. *Ans.* (*a*) $\pm\sqrt{21}\,m$; (*b*) $-4.88\,m$.

8. In Ex. 17, p. 348, find the momentum after 3 sec. *Ans.* 0.423 m

9. A body falls under gravity. Find the impulse during the time t.
Ans. mgt.

10. In Ex. 39, p. 338, find the impulse during the third second.
Ans. 3 m.

11. In Ex. 48, p. 338, find the impulse during the time t.

12. In Ex. 52, p. 338, find the impulse during the time t.

13. In Ex. 54, p. 338, find the impulse during the first minute.
Ans. $-3.03\,m$.

14. In Ex. 12, p. 340, find the impulse during the time t.
Ans. $mv_0\left(\frac{1}{1+kv_0t}-1\right)$.

15. In Ex. 27, p. 342, find the impulse during the time t.
Ans. $\frac{mp}{k^2}(e^{-kt}+kt-1)$.

16. In § 205, find the impulse during the passage from A to O.
Ans. $-mak$.

17. In Ex. 8, p. 348, find the impulse during the first 0.5 sec.
Ans. $-11.2\,m$.

18. In Ex. 35, p. 349, find the impulse during the first 2 seconds.
Ans. $-1.14\,m$.

19. A ship weighing 10,000 tons is moving with a velocity of 30 mi. per hr. What constant force is required to bring it to rest in 3 min.?
Ans. 77.4 tons.

20. A ball of mass $5\frac{1}{4}$ oz. strikes a bat with a velocity of $12\frac{1}{2}$ ft. per sec., and returns in the same line with a velocity of 32 ft. per sec. If the blow lasts $\frac{1}{20}$ sec., what average force is exerted by the batter?
Ans. 9 lbs.

211. Kinetic energy. By § 137, the *kinetic energy* of a particle of mass m moving with a velocity v is

$$E = \tfrac{1}{2}mv^2.$$

If a particle moves under given forces, the energy at any time can evidently be found from the (v, t)-equation; at any position, from the (v, x)-equation.

212. Work. When a particle moves in a straight line under the action of a *constant force* F, the *work* done is defined as the product of the force by the distance passed over.

When the force is variable, we proceed as follows: Take the line of motion as x-axis, and suppose the body moves from $x = a$ to $x = b$. Divide the interval into segments Δx, and multiply each segment Δx by the value of F at some point of Δx. The limit of the sum of the products thus obtained is defined as the *work* of the variable force during the motion:

$$W = \lim_{\Delta x \to 0} \sum F \, \Delta x = \int_a^b F \, dx.$$

Example: In Ex. 1, p. 340, find the work done by the resistance in any space interval.

Here

$$F = mj = - mkv.$$

But

$$v = v_0 - kx,$$

whence

$$W = - mk \int_0^x (v_0 - kx) dx = \tfrac{1}{2} mk^2x^2 - mkv_0x.$$

213. The principle of kinetic energy and work. Let us write the equation

$$m \frac{d^2x}{dt^2} = F$$

in the form (§ 189)

$$mv \frac{dv}{dx} = F.$$

Multiplying by dx and integrating between the x-limits x_1 and x_2 and the corresponding v-limits v_1 and v_2, we find

$$\tfrac{1}{2} mv_2^2 - \tfrac{1}{2} mv_1^2 = \int_{x_1}^{x_2} F \, dx.$$

The left member of this equation is the change in kinetic energy as the body moves from x_1 to x_2; the right member is the work done during the motion. Hence we have the *principle of kinetic energy and work:*

If a particle moves in a straight line, the change of kinetic energy in any space-interval is equal to the work done by the force in that interval.

EXERCISES

1. In Ex. 1, p. 335, find the kinetic energy (*a*) at the end of 2 sec.; (*b*) 1 ft. above the starting point. *Ans.* (*a*) 1352 *m*; (*b*) 40 *m*.

2. In Ex. 5, p. 335, find the kinetic energy (*a*) at the end of 2 sec.; (*b*) 20 ft. above the starting point. *Ans.* (*a*) 200 *m*; (*b*) 328 *m*.

3. In Ex. 37, p. 338, find the kinetic energy when $t = 2$.

Ans. 12.5 *m*.

4. In Ex. 20, p. 341, find the kinetic energy when $t = 0.1$.

Ans. 35.4 *m*.

5. In Ex. 23, p. 341, find the kinetic energy at the end of $\frac{1}{4}$ sec.

Ans. 94.5 *m*.

6. In Ex. 23, p. 345, find the kinetic energy when $t = \frac{\pi}{2k}$.

Ans. $\frac{p^2m}{2k^4}$.

7. A body falls under gravity through a distance x. Find the work done. *Ans.* mgx.

8. In Ex. 12, p. 340, find the work through the distance x.

Ans. $\frac{1}{2}mv_0^2(e^{-2kx} - 1)$.

9. In § 205, find the work done during the passage from A to O.

10. In Ex. 19, p. 345, find the work done during the first 2 ft.

Ans. 8 *m*.

11. A particle moves under the action of a force inversely proportional to the square of the distance from a fixed point O in the line of motion, and directed toward that point. Find the work done by the force in bringing the particle from infinity up to a distance a from the fixed point.

Ans. $\frac{mk}{a}$.

12. A freight train weighing 20,000 tons is running at 20 mi. per hr. What constant force is required to bring it to rest in $\frac{1}{4}$ mile?

Ans. 13.9 tons.

13. A ball of mass $5\frac{1}{4}$ oz. moving at 50 ft. per sec. is caught and brought to rest in a distance of 6 in. What is the average force on the hand? *Ans.* 26 lbs.

CHAPTER XXVIII

PLANE CURVILINEAR MOTION

214. Rotation. In discussing circular motion, it is usually convenient to take as dependent variable the angle θ swept out in the time t.

The problem of *uniformly accelerated* circular motion (§ 70) is closely analogous to that of uniformly accelerated rectilinear motion. The equation of motion is evidently

$$\frac{d^2\theta}{dt^2} = k,$$

where k is the constant angular acceleration.

215. The simple pendulum. A *simple pendulum* is a point swinging in a vertical circle under the acceleration of gravity.

Let P be a particle of mass m connected to the point O by a cord or rod of length l, and denote by θ the angle between OP and the vertical, by s the length of the arc AP. The effective force acting on P is the component of gravity tangent to the circle; since this is directed opposite to s, it must be given the minus sign. The equation of motion of P is therefore

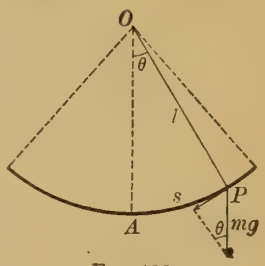


FIG. 108

$$\frac{d^2s}{dt^2} = -\,g \sin\theta. \tag{1}$$

But

$$s = l\theta,$$

so that (1) may be written

$$l\frac{d^2\theta}{dt^2} = -\,g \sin\theta. \tag{2}$$

A first integration of (2) can be performed by the method of § 189; the general solution, however, cannot be expressed in terms of elementary functions. We shall therefore consider only the case in which the oscillations are so small that $\sin\theta$ may be replaced by θ (cf. § 151), and (2) written in the form

$$(3) \qquad l\frac{d^2\theta}{dt^2} = -g\theta.$$

This equation shows that for small oscillations the motion is approximately simple harmonic. The remainder of the discussion is left to the student.

EXERCISES

1. Write and solve the equation of uniform circular motion, taking $\theta = 0$, $\omega = \omega_0$ when $t = 0$, where ω is the angular velocity.

2. Proceed as in Ex. 1 for uniformly accelerated circular motion.

3. A wheel is making 400 R. P. M. when resistance begins to retard its motion at the rate of 10 radians per sec. When will it come to rest? How many revolutions will it make before stopping?

Ans. $\frac{4}{3}\pi$ sec.; $\frac{40}{9}\pi$ rev.

4. Solve equation (3), § 215, taking $\theta = \theta_0$ and $\omega = 0$ when $t = 0$. For convenience, put $\frac{g}{l} = k^2$. *Ans.* $\theta = \theta_0 \cos kt$.

5. Show that, in the problem of the simple pendulum, the time of one *swing* or *beat* is

$$T = \pi\sqrt{\frac{l}{g}}.$$

6. Find the length of the "seconds pendulum" — *i.e.* a pendulum making one swing per second — at a place where $g = 32.17$ ft. per sec. per sec. *Ans.* 3.2595 ft.

7. Find the angular velocity ω in terms of θ if the oscillations are so large that (2), § 215, must be used.

8. For a pendulum 4 ft. long, within what range can equation (3) of § 215 be used, if the allowable error in the angular acceleration is 0.01? Take $g = 32$. *Ans.* $\theta < 11° 12'$.

9. Using Ex. 5, find approximately the change in the time of swing due to a small change in the length.

10. As the altitude above sea-level increases, the value of g diminishes. Using Ex. 5, find approximately the change in T due to a small change in g.

216. The equations of motion. In the general case of motion in a plane curve, it is convenient to resolve all the applied forces into components parallel to the coördinate axes. The product $m\frac{d^2x}{dt^2}$ of the mass by the x-component of the acceleration (see § 69) is equal to the sum F_x of the x-components of all the forces; similarly for the y-components. We thus have the *two equations of motion:*

$$m\frac{d^2x}{dt^2} = F_x, \quad m\frac{d^2y}{dt^2} = F_y.$$

In the most general case, both F_x and F_y are functions of x, y, t, and the velocity-components $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$. We shall, however, confine our attention to the case in which F_x is a function only of x, v_x, and t, and F_y is a function of y, v_y, and t. In this case the two equations of motion may be integrated separately. We thus obtain two equations giving respectively x and y in terms of t; these are parametric equations of the path of the moving point. By the same methods as those already used we find equations giving v_x and v_y in terms of t, and in terms of x and y respectively. The total velocity v is found by § 68.

217. Projectiles. A simple example of curvilinear motion is furnished by a projectile moving under gravity alone — *i.e.* in a medium whose resistance can be neglected.

Let a particle be projected with an initial velocity v_0 inclined at an angle α to the horizontal. With the starting point as origin and the y-axis positive upward, the initial conditions are

$$x = 0,\ y = 0,\ v_x = v_0\cos\alpha,\ v_0 = v_0\sin\alpha \text{ when } t = 0.$$

The force of gravity acts vertically downward; there is no horizontal force. Hence the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g.$$

These may be integrated and the constants determined precisely as in our earlier work.

EXERCISES

1. Solve the problem of § 217 completely, finding x, y, v_x, and v_y in terms of t, v_x in terms of x, and v_y in terms of y.

2. In Ex. 1, by eliminating t from the (x, t)- and (y, t)-equations, show that the path is a parabola.

3. Show that, in the ideal case of § 217, a projectile whose initial velocity is horizontal will strike the ground in the same time as a body let fall from rest from the same height.

4. The *time of flight* is the time from the starting point until the projectile strikes the ground. Show that on a horizontal plane the time of flight is

$$T = \frac{2\, v_0}{g} \sin \alpha.$$

5. The *range* of a projectile is the distance from the starting point to the point where it strikes the ground. Show that the range on a horizontal plane is

$$R = \frac{v_0^2}{g} \sin 2\,\alpha.$$

6. What elevation gives the greatest range on a horizontal plane?

7. A pitcher throws a ball with a speed of 100 ft. per sec., the ball leaving his hand horizontally at a height of 5 ft. Show that under the assumptions of § 217 the ball would strike the ground before reaching the batter 60 ft. away.

8. A stone is thrown horizontally from the top of a tower 400 ft. high, with a velocity of 20 ft. per sec. (*a*) When, (*b*) where, and (*c*) with what velocity does it strike the ground?

Ans. (*a*) 5 sec.; (*c*) 161.2 ft. per sec., at 7° 8′ to the vertical.

9. A man standing on the top of a cliff 300 ft. high throws a stone directly toward an object 200 ft. out from the foot of the cliff. If the stone leaves his hand with a velocity of 60 ft. per sec., by what distance does the stone miss the mark? *Ans.* 98.8 ft.

10. In Ex. 9, in what direction should the stone be thrown in order to reach the mark?

Ans. At either of the angles of elevation given by the equation $16 \tan^2 \alpha - 18 \tan \alpha - 11 = 0$.

11. A stone, starting from rest, slides down a roof sloping 30° to the horizontal, through a distance of 12 ft. If the lower edge of the roof is 50 ft. high, (*a*) when, (*b*) where, (*c*) with what velocity does the stone strike the ground?

Ans. (*a*) 2.72 sec.; (*b*) 25.2 ft. from the building; (*c*) 59.9 ft. per second, at 16° 28′ to the vertical.

12. From a train running 15 mi. per hr. a ball is thrown horizontally at right angles to the track, with a velocity of 22 ft. per sec. If it starts at a height of 8 ft., find when, where, and with what velocity the ball will strike the ground.

Ans. The ball moves in a plane making an angle of 45° with the track: $t = \frac{1}{2}\sqrt{2}$ sec., $x = 22$ ft., $v = 2\sqrt{370}$ ft. per sec., at 53° 58′ to the vertical.

13. From a balloon descending at the rate of 28 ft. per sec. a ball is thrown with a velocity of 40 ft. per sec. toward an object that is 40 ft. below the balloon and 30 ft. to the side. By what distance does the ball miss the mark? *Ans.* 16.1 ft.

14. Up an inclined plane 50 ft. long and 30 ft. high a ball is thrown with a velocity of 100 ft. per sec. When, where, and with what velocity will it strike the ground?

Ans. 4.38 sec.; 277 ft.; 100 ft. per sec., at 46° to the vertical.

15. A particle slides on a smooth roof inclined at 45° to the horizontal. If the initial velocity is 10 ft. per sec. parallel to the edge of the roof and the starting point is 20 ft. above the edge, find when, where, and with what velocity the particle leaves the roof.

Ans. $t = 1.33$ sec., $x = 13.3$ ft., $v = 31.7$ ft. per sec., at 48° to the vertical.

218. Central force proportional to the distance. A force that is always directed toward a fixed point is called a *central force*, the fixed point being the *center*. Such forces are of great importance in nature.

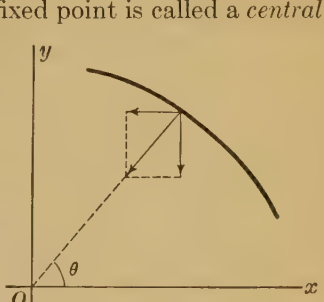


FIG. 109

We shall consider only the case in which the force is *proportional to the distance* from the center. Denoting the magnitude of the force by mk^2r, we find the cartesian components to be $-mk^2r\cos\theta$, $-mk^2r\sin\theta$. But

$$r\cos\theta = x, \quad r\sin\theta = y,$$

whence the components are $-mk^2x$, $-mk^2y$, and the equations of motion are

$$\frac{d^2x}{dt^2} = -k^2x, \quad \frac{d^2y}{dt^2} = -k^2y.$$

EXERCISES

1. A particle moves under the action of a force directed toward the origin O and proportional to the distance from O. If $x = 10$, $y = 0$, $v_x = 0$, $v_y = 20$ when $t = 0$, discuss the motion. Take $k = 1$.

2. Find the equation of the path in Ex. 1. *Ans.* $\dfrac{x^2}{100} + \dfrac{y^2}{400} = 1.$

3. Solve Ex. 1 if $x = 0$, $y = 2$, $v_x = 5$, $v_y = 0$ when $t = 0$.

4. Find the equation of the path in Ex. 3.

5. Solve the problem of § 218 with $k^2 = 4$, if $x = -3$, $y = 0$ when $t = 0$, $x = -\frac{3}{2}\sqrt{2}$, $y = \sqrt{2}$ when $t = \dfrac{\pi}{8}$.

6. Solve the problem of § 218 if the y-axis is vertical and gravity is to be considered. Assume $x = a$, $y = 0$, $v_x = 0$, $v_y = 0$ when $t = 0$.

Ans. $x = a \cos kt$, $y = \dfrac{g}{k^2}(\cos kt - 1)$.

7. Find the equation of the path in Ex. 6.

8. Solve Ex. 6 with the conditions $x = 0$, $y = a$, $v_x = v_0$, $v_y = 0$ when $t = 0$. *Ans.* $x = \dfrac{v_0}{k} \sin kt$, $y = \left(a + \dfrac{g}{k^2}\right) \cos kt - \dfrac{g}{k^2}$.

9. In Ex. 8, show that the path is an ellipse. Find its center and vertices.

INDEFINITE INTEGRALS

[In this table, integrals immediately reducible to a standard form are omitted.]

1. $\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\log\,(ax+b) + C.$

2. $\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\log\,(ax+b) + C.$

3. $\int x(ax+b)^n\,dx = \frac{x(ax+b)^{n+1}}{a(n+1)} - \frac{(ax+b)^{n+2}}{a^2(n+1)(n+2)} + C.$

4. $\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log\frac{x}{ax+b} + C.$

5. $\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\log\frac{x}{ax+b} + C.$

6. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} + C.$

7. $\int \frac{dx}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int\frac{dx}{ax^2+b}.$

8. $\int \frac{dx}{x(ax^2+b)} = \frac{1}{2b}\log\frac{x^2}{ax^2+b} + C.$

9. $\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2\arcsin\frac{x}{a} + C.$

10. $\int \sqrt{x^2\pm a^2}\,dx = \tfrac{1}{2}x\sqrt{x^2\pm a^2} \pm \tfrac{1}{2}a^2\log\,(x+\sqrt{x^2\pm a^2}) + C.$

11. $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \log\,(x+\sqrt{x^2\pm a^2}) + C.$

12. $\int \frac{dx}{x\sqrt{a^2\pm x^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2\pm x^2}} + C.$

13. $\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\arcsin\frac{a}{x} + C.$

14. $\int \frac{\sqrt{a^2\pm x^2}}{x}\,dx = \sqrt{a^2\pm x^2} + a\log\frac{x}{a+\sqrt{a^2\pm x^2}} + C.$

15. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} + a\arcsin\frac{a}{x} + C.$

16. $\int (a^2-x^2)^{\frac{3}{2}}\,dx = \tfrac{1}{4}x(a^2-x^2)^{\frac{3}{2}} + \tfrac{3}{8}a^2x\sqrt{a^2-x^2} + \tfrac{3}{8}a^4\arcsin\frac{x}{a} + C.$

17. $\int (x^2\pm a^2)^{\frac{3}{2}}\,dx = \tfrac{1}{4}x(x^2\pm a^2)^{\frac{3}{2}} \pm \tfrac{3}{8}a^2x\sqrt{x^2\pm a^2} + \tfrac{3}{8}a^4\log\,(x+\sqrt{x^2\pm a^2}) + C.$

18. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$

19. $\displaystyle\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C.$

20. $\displaystyle\int x^2\sqrt{a^2 - x^2}\,dx = -\tfrac{1}{4}x(a^2 - x^2)^{\frac{3}{2}} + \tfrac{1}{8}a^2x\sqrt{a^2 - x^2} + \tfrac{1}{8}a^4 \arcsin\frac{x}{a} + C.$

21. $\displaystyle\int x^2\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{4}x(x^2 \pm a^2)^{\frac{3}{2}} \mp \tfrac{1}{8}a^2x\sqrt{x^2 \pm a^2} - \tfrac{1}{8}a^4 \log\,(x + \sqrt{x^2 \pm a^2}) + C.$

22. $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\tfrac{1}{2}x\sqrt{a^2 - x^2} + \tfrac{1}{2}a^2 \arcsin\frac{x}{a} + C.$

23. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \tfrac{1}{2}x\sqrt{x^2 \pm a^2} \mp \tfrac{1}{2}a^2 \log\,(x + \sqrt{x^2 \pm a^2}) + C.$

24. $\displaystyle\int \frac{dx}{\sqrt{2\,ax - x^2}} = 2 \arcsin\sqrt{\frac{x}{2\,a}} + C.$

25. $\displaystyle\int \frac{x^n\,dx}{\sqrt{2\,ax - x^2}} = -\frac{x^{n-1}\sqrt{2\,ax - x^2}}{n} + \frac{a(2\,n - 1)}{n}\int \frac{x^{n-1}dx}{\sqrt{2\,ax - x^2}}.$

26. $\displaystyle\int \sqrt{2\,ax - x^2}\,dx = \tfrac{1}{2}(x - a)\sqrt{2\,ax - x^2} + \tfrac{1}{2}a^2 \arcsin\frac{x - a}{a} + C.$

27. $\displaystyle\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2\,x + C.$

28. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\,dx.$

29. $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2\,x + C.$

30. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n - 1}{n}\int \cos^{n-2} x\,dx.$

31. $\displaystyle\int \cos^m x\sin^n x\,dx = \frac{\cos^{m-1} x\sin^{n+1} x}{m + n} + \frac{m - 1}{m + n}\int \cos^{m-2} x\sin^n x\,dx.$

32. $\displaystyle\int \cos^m x\sin^n x\,dx = -\frac{\sin^{n-1} x\cos^{m+1} x}{m + n} + \frac{n - 1}{m + n}\int \cos^m x\sin^{n-2} x\,dx.$

33. $\displaystyle\int \tan x\,dx = -\log\cos x + C.$

34. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n - 1} - \int \tan^{n-2} x\,dx.$

35. $\displaystyle\int \cot x\,dx = \log\sin x + C.$

36. $\displaystyle\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n - 1} - \int \cot^{n-2} x\,dx.$

37. $\displaystyle\int \sec x\,dx = \frac{1}{2}\log\frac{1 + \sin x}{1 - \sin x} + C.$

38. $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx.$

39. $\int \csc x\,dx = \log \tan \tfrac{1}{2} x + C.$

40. $\int \csc^n x\,dx = -\dfrac{\cot x \csc^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\int \csc^{n-2} x\,dx.$

41. $\int x \sin x\,dx = \sin x - x \cos x + C.$

42. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx.$

43. $\int x \cos x\,dx = \cos x + x \sin x + C.$

44. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx.$

45. $\int x \sin^n x\,dx = \dfrac{\sin^{n-1} x\,(\sin x - nx \cos x)}{n^2} + \dfrac{n-1}{n}\int x \sin^{n-2} x\,dx.$

46. $\int x \cos^n x\,dx = \dfrac{\cos^{n-1} x\,(\cos x + nx \sin x)}{n^2} + \dfrac{n-1}{n}\int x \cos^{n-2} x\,dx.$

47. $\int xe^{ax}\,dx = \dfrac{e^{ax}}{a^2}(ax - 1) + C.$

48. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1} e^{ax}\,dx.$

49. $\int \sinh x\,dx = \cosh x + C.$

50. $\int \cosh x\,dx = \sinh x + C.$

51. $\int \tanh x\,dx = \log \cosh x + C.$

52. $\int \sinh^2 x\,dx = \tfrac{1}{2} \sinh x \cosh x - \tfrac{1}{2} x + C.$

53. $\int \cosh^2 x\,dx = \tfrac{1}{2} \sinh x \cosh x + \tfrac{1}{2} x + C.$

54. $\int e^{ax} \sin mx\,dx = \dfrac{e^{ax}(a \sin mx - m \cos mx)}{m^2 + a^2} + C.$

55. $\int e^{ax} \cos mx\,dx = \dfrac{e^{ax}(m \sin mx + a \cos mx)}{m^2 + a^2} + C.$

56. $\int \log x\,dx = x \log x - x + C.$

57. $\int x^n \log x\,dx = x^{n+1}\left[\dfrac{\log x}{n+1} - \dfrac{1}{(n+1)^2}\right] + C.$

INDEX

(The references are to pages.)

Acceleration, 108, 331
  angular —, 113, 355
  in curvilinear motion, 110, 357
  of gravity, 108, 333
Algebraic functions, 3, 19
Approximate formulas, 92, 256, 261, 270
Areas (plane), 171, 176, 179
  in polar coördinates, 183
Asymptotes, 120
Atwood's machine, 334

Catenary, 82, 181, 189, 196, 199
Centroid, 204
  of arcs, 214
  of areas, 209
  of surfaces, 214
  of volumes, 212
Characteristic equation, 319
Concavity, 38
Conjugate point, 129
Continuity, 10
  of a function of two variables, 266
Critical point, 37
Curvature, 100
  center of —, 102
  circle of —, 102
  radius of —, 101
Curve tracing, 40, 121
  by composition of ordinates, 56
  in polar coördinates, 131
Cusp, 129
Cycloid, 182, 190, 197, 199

Definite integral, 165, 188
  change of variable in —, 167
  fundamental theorem for —, 178
  geometric meaning of —, 171
Density, 203, 290
Derivative, 14, 90
  geometric meaning of —, 15
  higher —, 26, 97
Derived curves, 47, 108
Differential, 89
  of arc, 99
  total —, 269
Differential equation, 298
  exact —, 306
  general solution of —, 300, 313
  linear —, 309, 310, 318, 325, 327
  order of —, 298
  particular solution of —, 300, 313
Differentiation, 14
  of algebraic functions, 19
  of exponential functions, 79
  of implicit functions, 30, 270
  of inverse trigonometric functions, 65
  of logarithms, 76
  of trigonometric functions, 51, 52
  standard formulas of —, 83–84
Double integrals, 280
  in polar coördinates, 284
  transformation of —, 288, 289
Double point, 128

Equation of motion, 332, 357
Error, 92, 241, 270, 272
Exponential equations, 74
Exponential function, 68, 321

Fluid pressure, 200
Force, 199, 332, 357
  center of —, 342, 359
  central —, 359
Functions, 1
  branches of —, 4
  continuous —, 9
  differentiable —, 15
  homogeneous —, 305
  increasing and decreasing —, 37
  integrable —, 171, 179
  inverse —, 60
  kinds of —, 3
  of several variables, 265
  one-valued —, 4

Geometric addition, 110
Geometric derivative, 112
Graphic solution of equations, 57

Heterogeneous masses, 290
Hooke's law, 343
Hyperbolic functions, 81
Hypocycloid, 182, 190, 196, 198

Implicit functions, 29, 270
Improper integrals, 172
Impulse, 350
Indefinite integral, 132
  geometric meaning of —, 171
Indeterminate forms, 116
Infinitesimals, 85, 187, 269
  order of —, 85
  principal part of —, 86, 269
Infinity, 11, 12
Integral curves, 300
Integral table, 361–363
Integrating factor, 307, 309
Integration, 132
  by parts, 143
  by substitution, 152
  of rational fractions, 157
  standard formulas of —, 136–137
Inverse trigonometric equations, 62
Inverse trigonometric functions, 61
Isolated point, 129

Kinetic energy, 225, 353

Law of the mean, 115
Length of arc, 195
Limit, 6, 7, 11, 12
  evaluation of —, 118
  of function of two variables, 266
  of sin α/α, 51, 248
  theorems on —, 8, 234
Logarithmic equations, 74
Logarithms, 68
  change of base of —, 70
  computation of —, 257
  natural —, 71

Maxima and minima, 37, 42, 58
Moment
  of inertia, 215, 216, 221, 223
  of mass, 203
Momentum, 350
Motion
  circular —, 113, 355
  curvilinear —, 110, 355
  in a resisting medium, 338
  on an inclined plane, 336
  rectilinear —, 107, 331
  simple harmonic —, 342, 356
  uniform —, 107
  uniformly accelerated —, 107, 333, 355

Node, 129
Normal, 33
  to a surface, 275

Pappus, propositions of, 211, 214
Parametric equations, 96, 169
Partial derivatives, 266, 267
  geometric meaning of —, 267
Partial fractions, 157
Pendulum, simple, 355
Point of inflection, 40
Projectiles, 357

Radius of gyration, 215
Radius of inertia, 215
Rate of change, 6, 15, 104, 108
  of derivative, 38
Rectification of curves, 195
Rotation, 113, 355

Series, 228
  absolute convergence of —, 239
  alternating —, 238
  computation by —, 248, 253, 256, 257
  convergence of —, 230, 232, 234, 236
  Maclaurin's —, 246
  power —, 243
  sum of —, 229, 240
  Taylor's —, 261
  transformation of —, 252
Singular points. 126
Slope of curve, 6, 15, 37
Surfaces
  angle between lines and —, 276
  angle between —, 276
  of revolution, 197
Symmetry, 121

Tangent, 33
Tangent plane, 274
Time-rates, 104
Trigonometric functions, 50
Trigonometric integrals, 188
Trigonometric substitutions, 155
Triple integrals, 294
Triple point, 128

Vectors, 110
Velocity, 107, 331
  angular —, 113
  in curvilinear motion, 110, 357
Vibrations, damped, 346
Volumes, 191, 278, 285
  of revolution, 184, 186, 287

Work, 352

$dm\,\bar{y} = \int y\,dm$ where (x, y, z) is the centroid of dm.

$dm\,\bar{z} = \int z\,dm$

Rule

1st. find dm

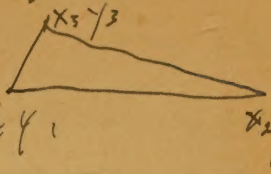
2nd. find centroid of dm

3rd. substitute in formulas.

Centroid of any triangle

$\frac{1}{3}(x_1 + x_2 + x_3)$

$\frac{1}{3}(y_1 + y_2 + y_3)$



I rod, rect to one end $\frac{1}{3}Ma^2$

rectangle rect. to base

cylinder rect to base

rect ∥ opp'd rect to base

prism rect to base

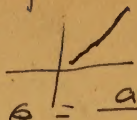
I circular disc rect to center $\frac{1}{2}Ma^2$

circular cylinder ∥ to axis

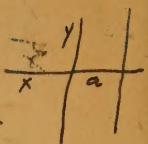
I triangle rect to base $\frac{1}{6}Mh^2$

I vol. of sphere to diameter $\frac{2}{5}Ma^2$

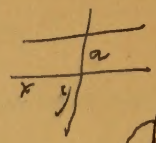
$\theta = a$



$r = a \sec\theta = \frac{a}{\cos\theta}$, $r\cos\theta = a$, $x = a$

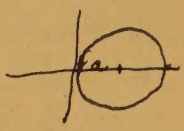


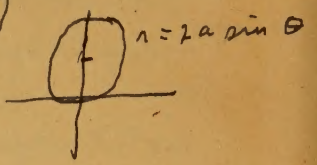
$r = a \operatorname{cosec}\theta$, $y = a$



$r = 2a\cos\theta$

$r = 2a\sin\theta$

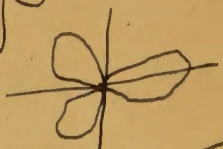




$r = a\cos 2\theta$

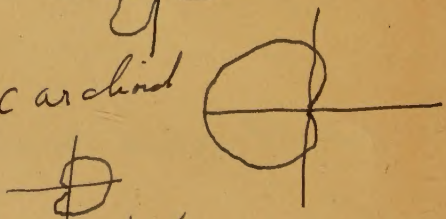
$r = a\sin 2\theta$

$r = a\cos^3\theta$

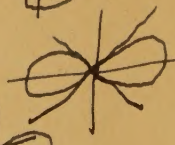


$r = a(1 - \cos\theta)$, Cardioid

$r = a(1 + \cos\theta)$

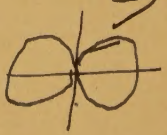


$r^2 = a^2\cos 2\theta$



Lemniscate

$r^2 = a^2\cos\theta$



$x = r\cos\theta$

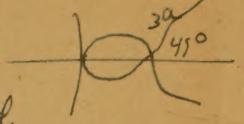
$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$

helicoid = circular stair

$4ay^2 = x(x-3a)^2$



1. Find the area of the loop / ret. plan. pole at $x = 3a$

2. Find centroid of one half area of loop. Moment of inertia, with respect to x or y axis

---

(2) Volume of solid of revolution about x and y axis of $x = 3a$.

Find centroids

Moments of inertia, axis of revolution, planes ⊥ to axis

(3) Cylinder on $4ay^2 = x(x-3a)^2$

Slice ~~$x = x$~~ $y = y$ $x = 3$

(4) Find length of loop

centroid of arc above x axis

$\bar{I}$ of arc with respect to x axis or y axis

(5) Surface of revolution, x axis, y axis or $x = 3a$

centroid

moment of inertia.

$\theta = xy$